# Documentation for the anisotropic/asymmetric energy degradation element

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**Abstract**

Present documentation summarizes the implementation details of an anisotropic energy degradation model for the phase-field finite element used to simulate brittle fracture. More about the phase-field method can be found in our recent article [1]. The energy split used in this implementation is based on the work of Moës et al. [2]. The material model is used in a 2D plane strain case.

## 1. Theory

The plus (+) and minus (-) signs in the upper index indicates positive (tension) and negative (compression) properties. To distinguish between compression and tension degradation the potential energy is divided into two part:

$$\psi = g\left(d\right)\psi_0^+ + \psi_0^-. \tag{1}$$

$g(d)$ is the degradation function: $g\left(d\right) = \left(1-d\right)^2$. $d$ is the damage variable varying from 0 to 1, representing undamaged and damaged materials respectively. The positive and negative energies on the left side can be written as a function of the principal strains:

$$\psi_0^{\pm} = \frac{E\nu}{2\left(1+\nu\right)\left(1-2\nu\right)}\left\langle tr\left(\boldsymbol{\varepsilon}\right)\right\rangle_{\pm}^2 + \frac{E}{2\left(1+\nu\right)}\left(\left\langle\varepsilon_2\right\rangle_{\pm}^2 + \left\langle\varepsilon_2\right\rangle_{\pm}^2\right), \tag{2}$$

where $E$ is Young's modulus, $\nu$ is Poisson's ratio. The first part of equation (2) represents the volumetric strains with the trace of the strain tensor: $tr\left(\boldsymbol{\varepsilon}\right) = \varepsilon_1 + \varepsilon_2$, while the second term account for deviatoric degradations. Functions $\left\langle\right\rangle_{\pm}$ stand for: positive $\left\langle x\right\rangle_{+} = \left(x + \left|x\right|\right)/2$ and negative $\left\langle x\right\rangle_{-} = \left(x - \left|x\right|\right)/2$ part. $\varepsilon_{1,2}$ are the principal strains.

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After substituting eq. (2) into (1) and taking its derivatives respect to the principal strains we obtain the following equation system:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \frac{E}{1+\nu} \begin{bmatrix} (1+\alpha_1 d)^2 + \gamma(1+\alpha d)^2 & \gamma(1+\alpha d)^2 \\ \gamma(1+\alpha d)^2 & (1+\alpha_2 d)^2 + \gamma(1+\alpha d)^2 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}. \tag{3}$$

In eq. (3) $\gamma = \nu/(1-2\nu)$ and the coefficients $\alpha$, $\alpha_1$ and $\alpha_2$ are taking the positive and negative parts in eq. (2) into account:

$$\begin{array}{cc} \alpha_i = 0 & \text{if} \rightarrow \varepsilon_i < 0 \\ \alpha_i = 1 & \text{if} \rightarrow \varepsilon_i \geq 0 \\ \alpha = 0 & \text{if} \rightarrow \varepsilon_1 + \varepsilon_2 < 0 \\ \alpha = 1 & \text{else} \rightarrow \varepsilon_1 + \varepsilon_2 \geq 0 \end{array} \tag{4}$$

Eq. (3) can be written is a short form as: $\hat{\boldsymbol{\sigma}} = \boldsymbol{L}\hat{\boldsymbol{\varepsilon}}$.

To obtain the stresses from the energy function, its derivative shell be calculated respect to the strain tensor:

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} = \frac{\partial \psi}{\partial \hat{\boldsymbol{\varepsilon}}} \frac{\partial \hat{\boldsymbol{\varepsilon}}}{\partial \boldsymbol{\varepsilon}} = \hat{\boldsymbol{\varepsilon}}^T \mathbf{L} \frac{\partial \hat{\boldsymbol{\varepsilon}}}{\partial \boldsymbol{\varepsilon}}. \tag{5}$$

In this implementation the strain gradient is calculated using a finite difference approximation. For example, its first term reads as:

$$\frac{\partial \varepsilon_1}{\partial \varepsilon_x} \simeq \frac{\varepsilon_1 (\varepsilon_x + d\varepsilon_x) - \varepsilon_1 (\varepsilon_x)}{d\varepsilon_x}, \tag{6}$$

where the principal strain ($\varepsilon_1$) is recalculated with the addition of an infinitesimal increment ($d\varepsilon_x$), and the difference is divided by $d\varepsilon_x$. The advantage of this method compared to the analytic solution presented in Ref. [2], is that this way derivatives can be determined always, while the analytic solution is discontinuous.

To obtain the materials stiffness matrix, the stress is differentiated respect to the strains:

$$\mathbf{C} = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} = \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon}^2} = \left( \frac{\partial \hat{\boldsymbol{\varepsilon}}}{\partial \boldsymbol{\varepsilon}} \right)^T \mathbf{L} \frac{\partial \hat{\boldsymbol{\varepsilon}}}{\partial \boldsymbol{\varepsilon}} + \hat{\boldsymbol{\varepsilon}}^T \mathbf{L} \frac{\partial^2 \hat{\boldsymbol{\varepsilon}}}{\partial \boldsymbol{\varepsilon}^2}. \tag{7}$$

The second derivatives as well are calculated using finite difference approximations. The diagonal terms can be expressed as:

$$\frac{\partial^2 \varepsilon_1}{\partial \varepsilon_x^2} \simeq \frac{\varepsilon_1 (\varepsilon_x + d\varepsilon_x) - 2\varepsilon_1 (\varepsilon_x) + \varepsilon_1 (\varepsilon_x - d\varepsilon_x)}{d\varepsilon_x^2}. \tag{8}$$

| Variable | Number of SDV in Abaqus |
|---|---|
| displacement - $u_x, u_y$ | SDV1-SDV2 |
| axial strains - $\epsilon_x, \epsilon_y$ | SDV3-SDV4 |
| engineering shear strain - $\gamma_{xy}$ | SDV5 |
| principal strain - $\epsilon_1, \epsilon_2$ | SDV6-SDV7 |
| volumetric strain - $\epsilon_1 + \epsilon_2$ | SDV8 |
| axial stress - $\sigma_x, \sigma_y$ | SDV9-SDV10 |
| shear stress - $\tau_{xy}$ | SDV11 |
| strain energy - $\psi$ | SDV12 |
| elastic strain energy - $\psi_0$ | SDV13 |
| phase-field - $d$ | SDV14 |
| Phase-field element | |
| phase-field - $d$ | SDV15 |
| history field - $H$ | SDV16 |

Table 1: Solution dependent variables used to plot the results.

While the of-diagonal components as:

$$\frac{\partial^2 \varepsilon_1}{\partial \varepsilon_x \partial \varepsilon_y} \simeq \frac{\begin{array}{c}\varepsilon_1\left(\varepsilon_x + d\varepsilon_x, \varepsilon_y + d\varepsilon_y\right) - \varepsilon_1\left(\varepsilon_x + d\varepsilon_x, \varepsilon_y - d\varepsilon_y\right) - \\ - \varepsilon_1\left(\varepsilon_x - d\varepsilon_x, \varepsilon_y + d\varepsilon_y\right) + \varepsilon_1\left(\varepsilon_x - d\varepsilon_x, \varepsilon_y - d\varepsilon_y\right)\end{array}}{4 d\varepsilon_x d\varepsilon_y}. \tag{9}$$

Due to the linear relationship between stress and strain eq. (5) is equivalent to:

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}. \tag{10}$$

To obtain a stable calculation the materials stiffness is only updated in the first internal iteration step. Then eq. (10) is used to determine the changing stress values due to strain redistribution. This step probably introduces some numerical errors, but linearizes the calculation, which makes it incredibly robust.

A tutorial dedicated to analyze the effect of the asymmetric energy degradation can be found on the following link: `http://www.molnar-research.com/tutorials_videos_4.html`

## 2. Solution dependent variables (SDV)

Tab. 1 summarizes the concerning mechanical parameters (displacement, strain, stress, etc.) for each solution dependent variable (SDV).

## References

[1] G. Molnár, A. Gravouil, Abaqus implementation of a robust staggered phase-field solution to model brittle fracture, Finite Element in Analysis and Design 130 (2017) 27–38.

[2] N. Moës, C. Stolz, P.-E. Bernard, N. Chevaugeon, A level set based model for damage growth: The thick level set approach, International Journal for Numerical Methods in Engineering 86 (3) (2011) 358–380.