



Fracture modelling

Gergely Molnár



Motivation

Why Modeling?



- ☐ Experiment are expensive
- ☐ Postprocessing is difficult
- ☐ Dangerous
- ☐ Difficult to design
- ☐ **BUT NECESSARY!!!!**

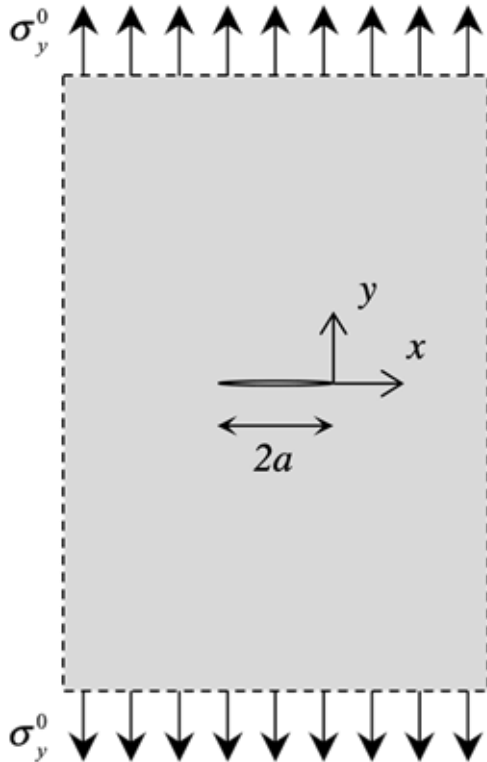


Stress intensity factor

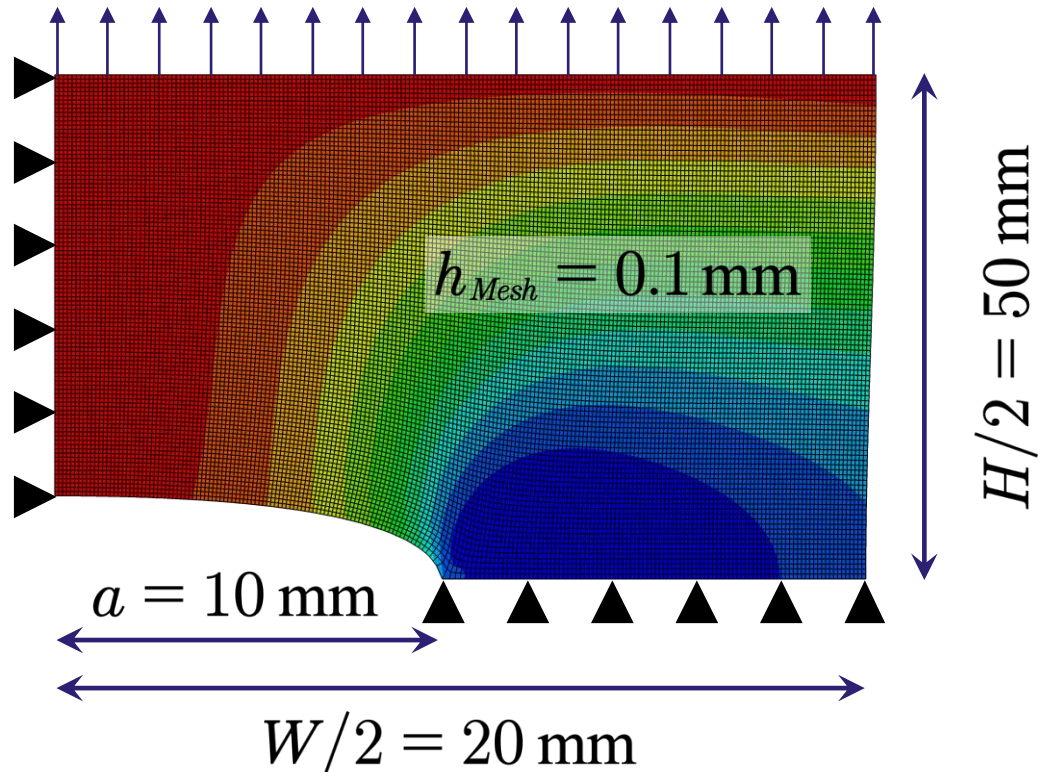
$$\mu = 70 \text{ GPa}$$

$$\nu = 0.21$$

Numerical example



$$\sigma_y^0 = 5 \text{ GPa}$$



Stress intensity factor

Near crack tip solution

$$u_x(r, \theta) = \frac{K_I}{8\mu\pi} \sqrt{2\pi r} \left[(2\kappa - 1) \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \right]$$

$$u_y(r, \theta) = \frac{K_I}{8\mu\pi} \sqrt{2\pi r} \left[(2\kappa + 1) \sin \frac{\theta}{2} - \sin \frac{3\theta}{2} \right]$$

μ shear modulus

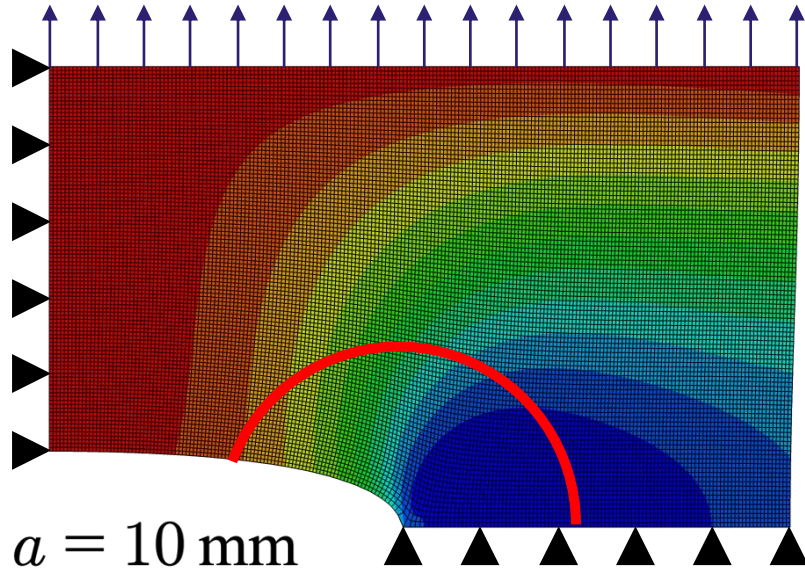
$$\kappa = \begin{cases} 3 - 4\nu & \text{plane strain} \\ \frac{3 - \nu}{1 + \nu} & \text{plane stress} \end{cases}$$



Stress intensity factor

Near crack tip solution

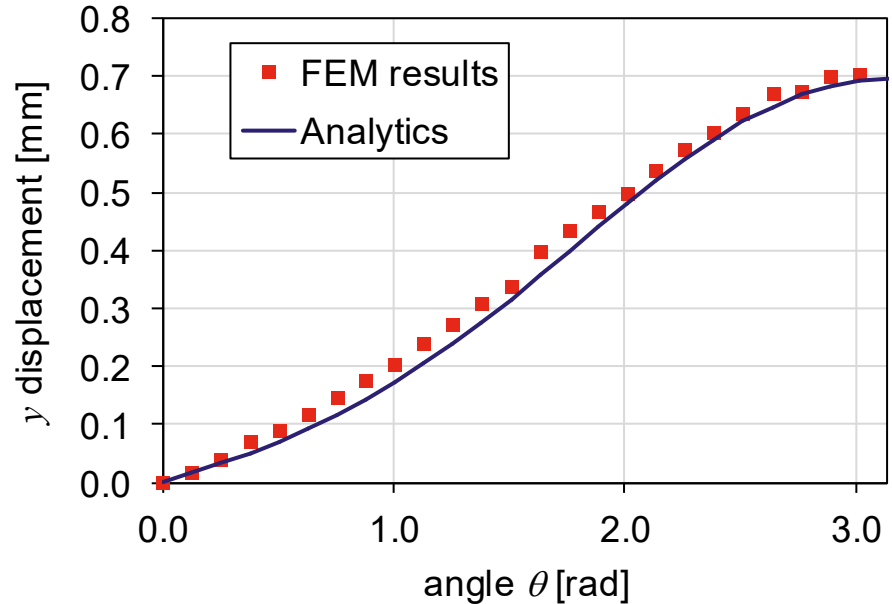
$$\sigma_y^0 = 5 \text{ GPa}$$



$$a = 10 \text{ mm}$$

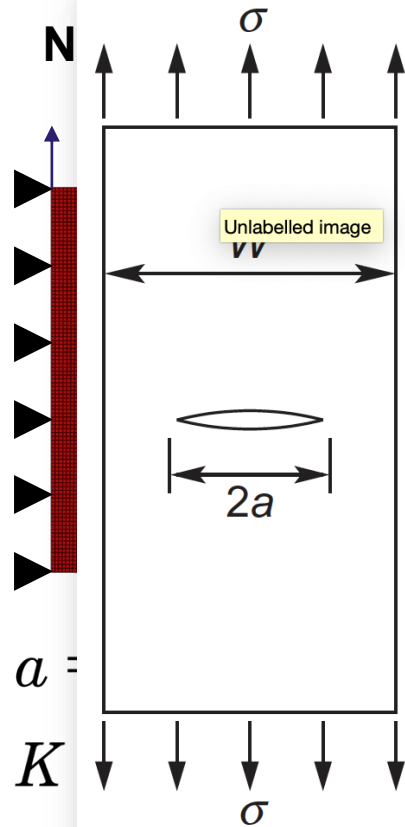
$$K = \sigma_y^0 \sqrt{a\pi}$$

$$= 28.02 \text{ GPa}\sqrt{\text{mm}}$$



$$K_{fit} = 31.90 \text{ GPa}\sqrt{\text{mm}}$$

Stress intensity factor



$$K_I = \sigma \sqrt{\pi a} F(\alpha), \quad \alpha = 2a/W$$

$$F(\alpha) = \sqrt{\sec(\pi \alpha / 2)}$$

Accuracy: 0.3% for $\alpha \leq 0.7$

$$F(\alpha) = 1 + 0.128\alpha - 0.288\alpha^2 + 1.525\alpha^3$$

Accuracy: 0.5% for $\alpha \leq 0.7$

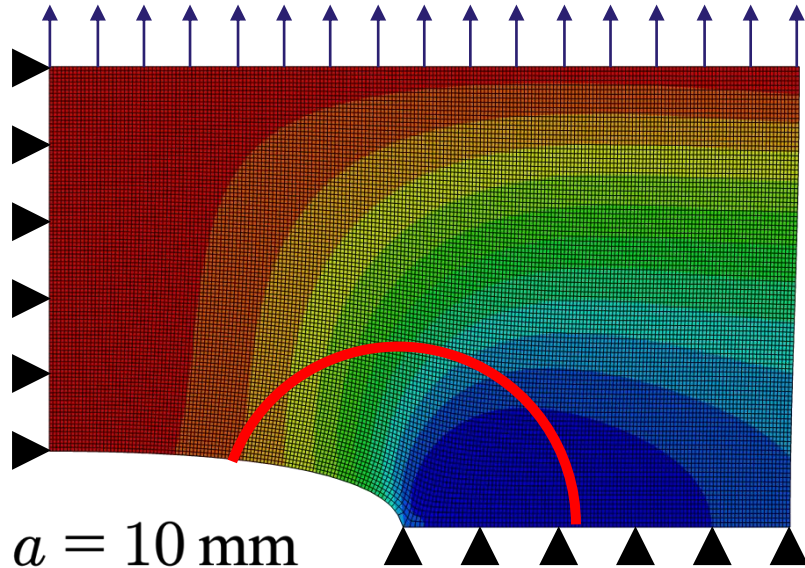
$$F(\alpha) = (1 - 0.5\alpha + 0.37\alpha^2 - 0.044\alpha^3) / \sqrt{1 - \alpha}$$

Accuracy: 0.3% for any α

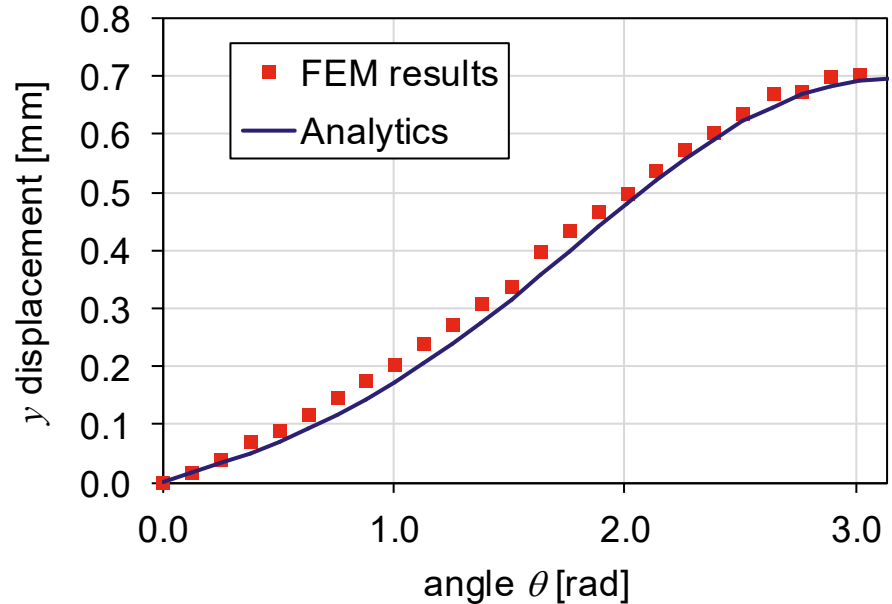
Stress intensity factor

Near crack tip solution

$$\sigma_y^0 = 5 \text{ GPa}$$



$$K = \sigma_y^0 \sqrt{a\pi} F(\alpha)$$
$$= 33.33 \text{ GPa}\sqrt{\text{mm}}$$

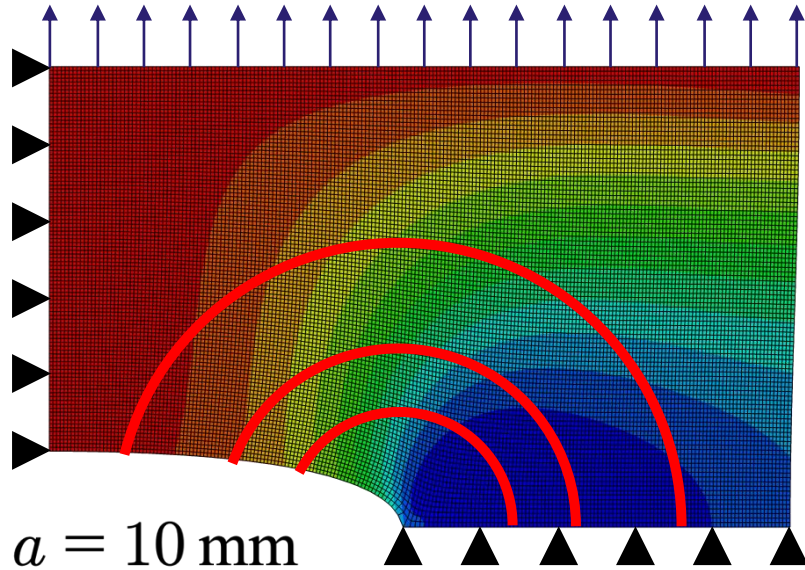


$$K_{fit} = 31.90 \text{ GPa}\sqrt{\text{mm}}$$

Stress intensity factor

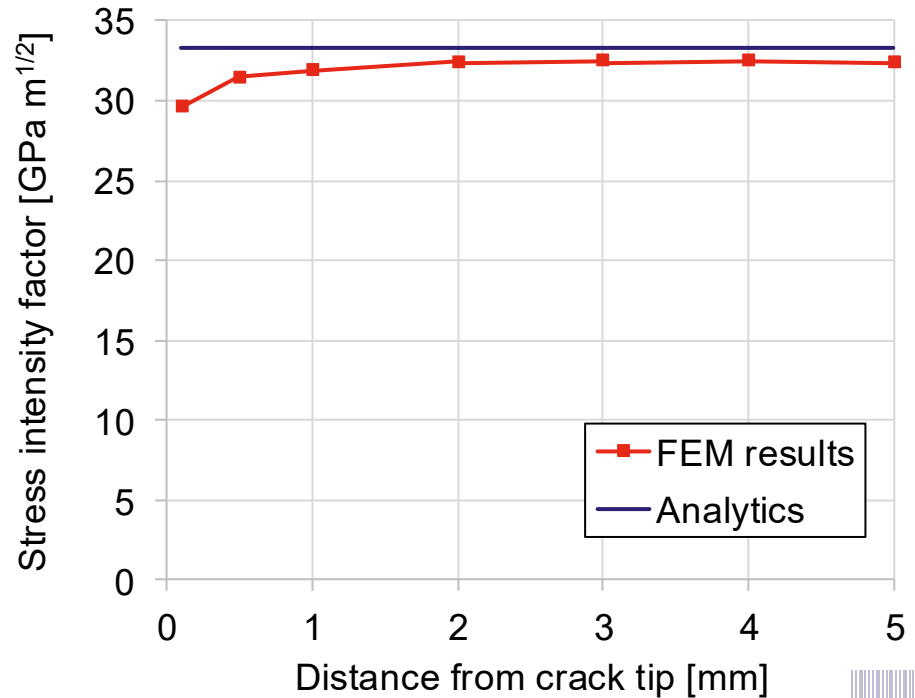
Near crack tip solution

$$\sigma_y^0 = 5 \text{ GPa}$$



$$a = 10 \text{ mm}$$

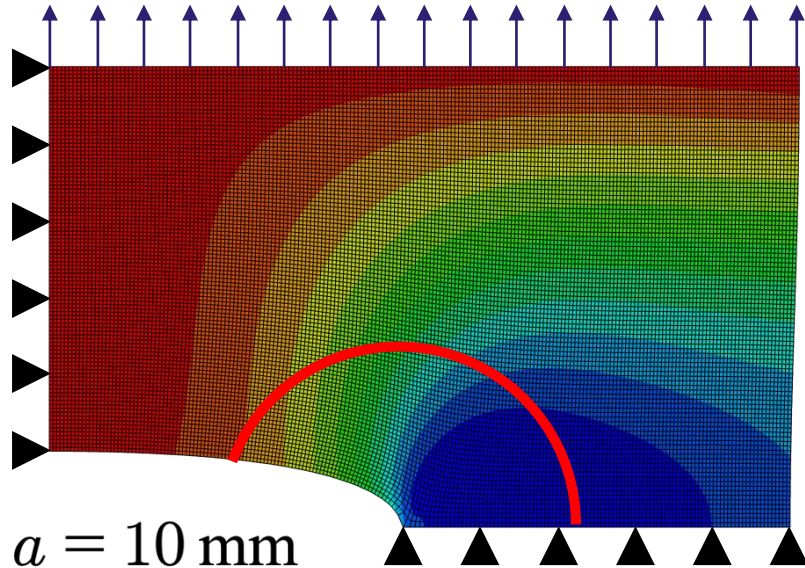
$$K = \sigma_y^0 \sqrt{a\pi} F(\alpha)$$
$$= 33.33 \text{ GPa}\sqrt{\text{mm}}$$



Stress intensity factor

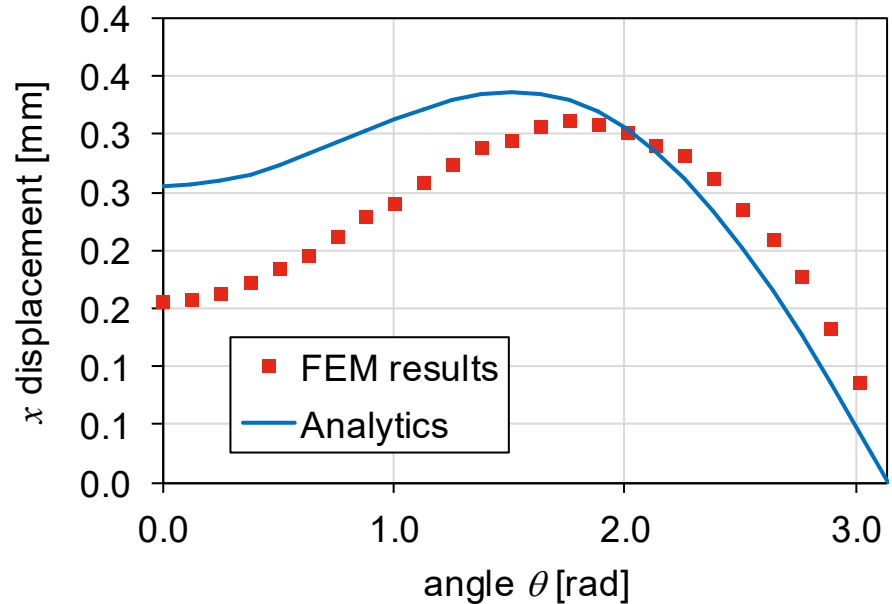
Near crack tip solution

$$\sigma_y^0 = 5 \text{ GPa}$$



$a = 10 \text{ mm}$

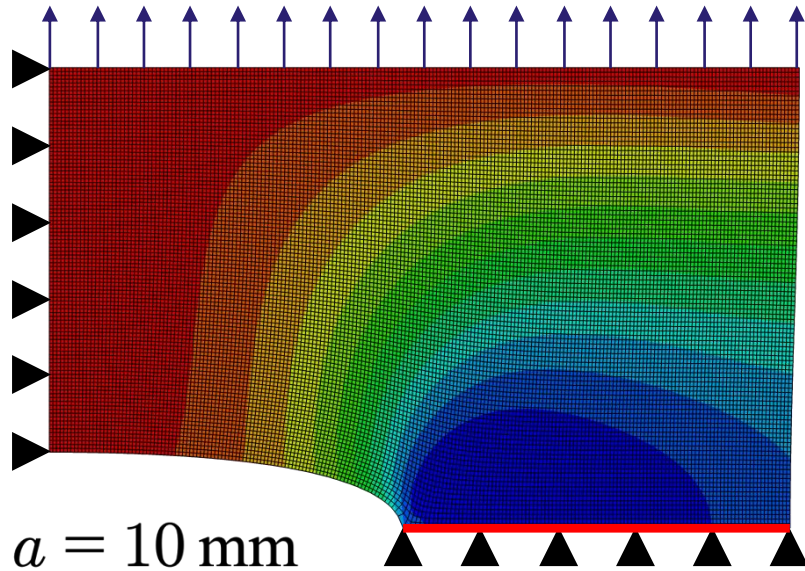
$$K = \sigma_y^0 \sqrt{a\pi} F(\alpha)$$
$$= 33.33 \text{ GPa}\sqrt{\text{mm}}$$



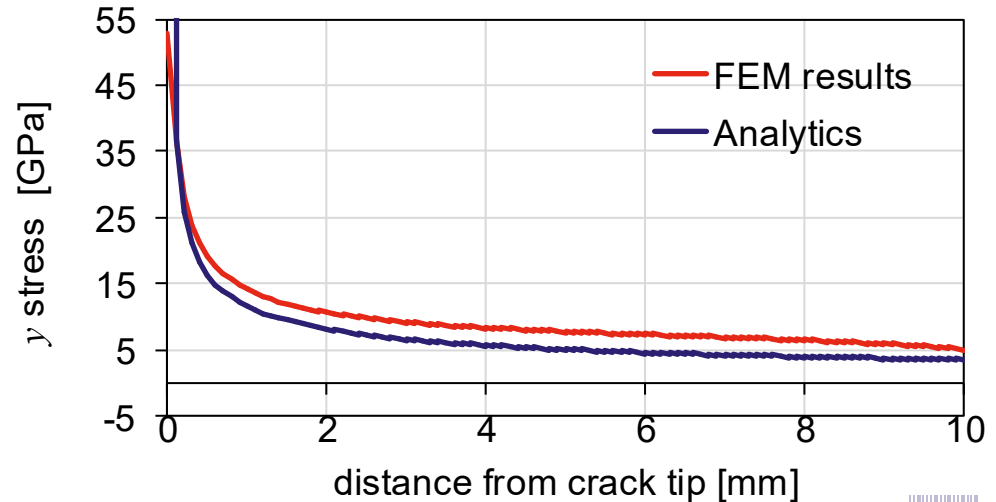
Stress intensity factor

Near crack tip solution

$$\sigma_y^0 = 5 \text{ GPa}$$

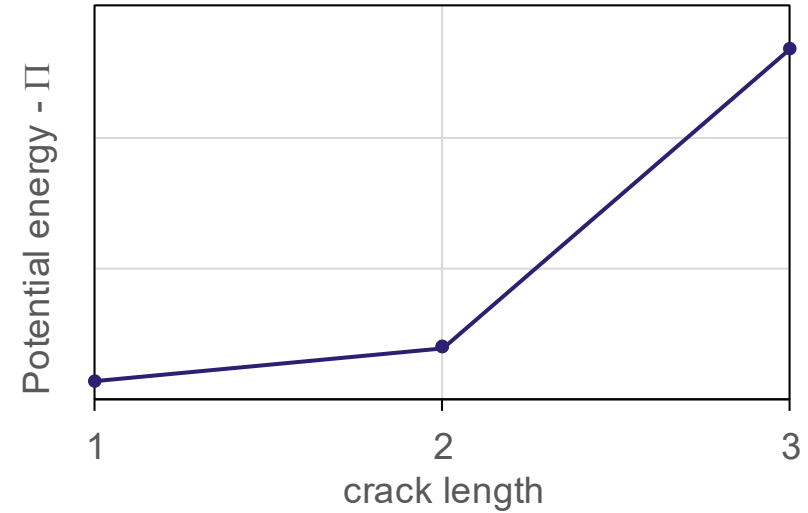
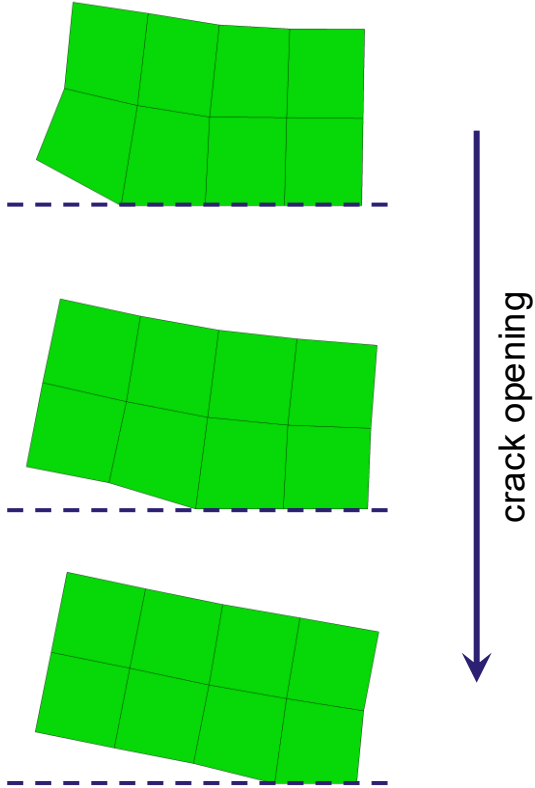


$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 + \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right]$$



Energy release rate

Basic concept



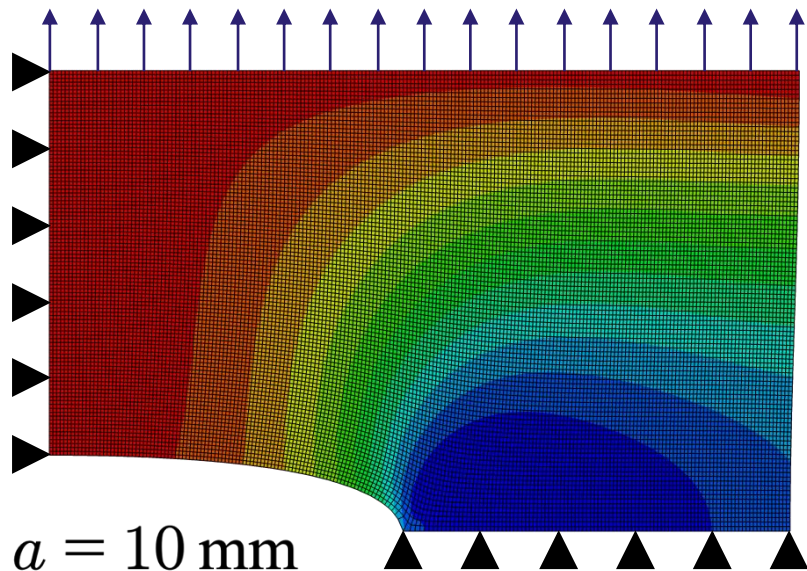
$$G\left(a + \frac{\Delta a}{2}\right) = -\frac{\Pi(a + \Delta a, P) - \Pi(a, P) - \Delta W_{ext}(P)}{\Delta a}$$

$$K = \sqrt{\frac{GE}{(1 - \nu^2)}}$$

Energy release rate

Numerical example

$$\sigma_y^0 = 5 \text{ GPa}$$



$$K = \sigma_y^0 \sqrt{a\pi} F(\alpha)$$

$$= 33.33 \text{ GPa}\sqrt{\text{mm}}$$

$$K = \sqrt{\frac{GE}{(1-\nu^2)}} = 33.28 \text{ GPa}\sqrt{\text{mm}}$$

$$\Delta a = 0.1 \text{ mm}$$

$$a = 10 - \frac{0.1}{2} = 9.95 \text{ mm}$$

Results

$$\Delta \Pi = 1.5080 \text{ J}$$

$$\Delta W_{ext} = 3.0160 \text{ J}$$

$$G = 15.12 \frac{\text{J}}{\text{mm}}$$

J-integral

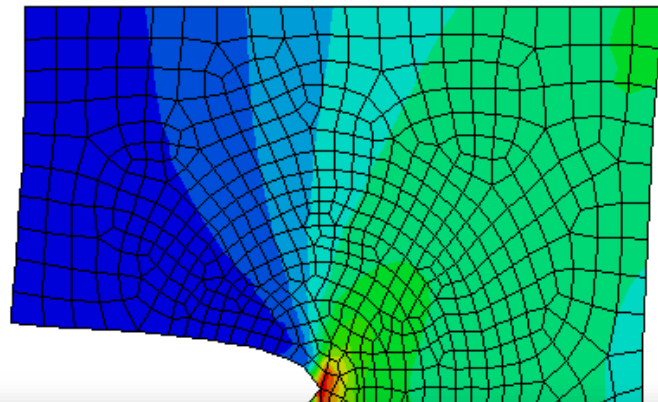
Abaqus

$$J_i = \int_{\Gamma} \pi n_i - t \frac{\partial u}{\partial x_i} d\Gamma$$

$$\pi = \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij}$$

$$t = \sigma n$$

$$K = \sqrt{\frac{JE}{(1-\nu^2)}}$$

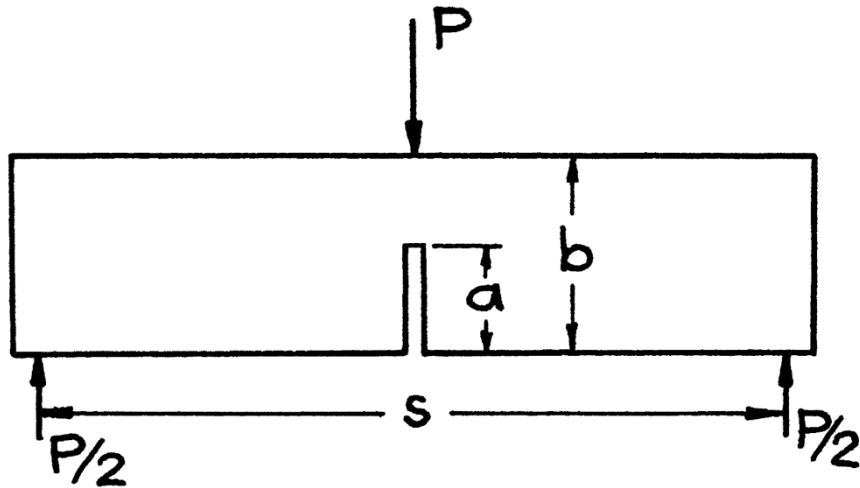


CRACK NAME	CRACKFRONT NODE SET	C O N T O U R S				
		1	2	3	4	5
H-OUTPUT-1_CRACK-1						
	-6-	K1: 20.10	24.03	23.95	23.99	24.00
		K2: -0.7783	2.3292E-02	3.5100E-02	-2.3945E-03	2.5224E-03
MERR	DIRECTION (DEG):	5.148	0.000	0.000	0.000	0.000
	J from Ks:	5.523	7.886	7.830	7.857	7.864



Tutorial:

<https://www.youtube.com/watch?v=ztrW-Gg5Qts>



$$\sigma = \frac{6M}{b^2}$$

$$M = \frac{Ps}{4}$$

$$K_I = \sigma \sqrt{\pi a} F\left(\frac{a}{b}\right)$$

- 1) Analytical verification
 - a) Displacement field
 - b) Energy release rate
 - c) Effect of initial crack-tip roundness

$$s = 40 \text{ mm}$$

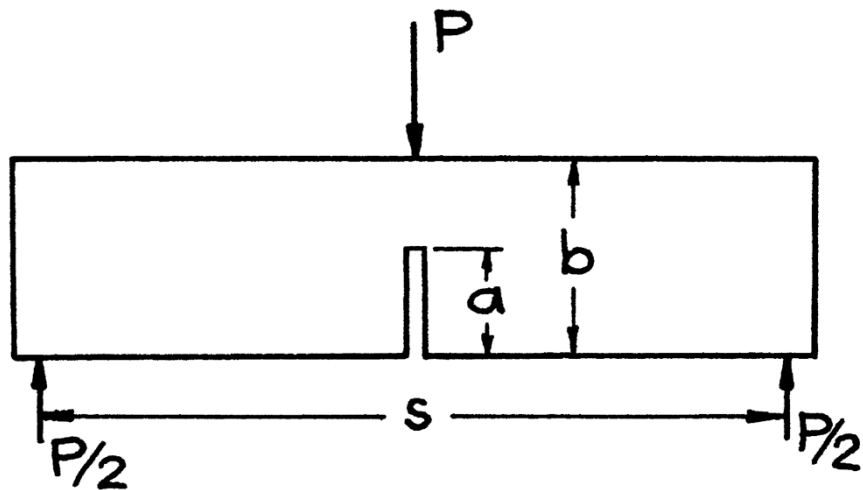
$$b = 5 \text{ mm}$$

$$P = 2 \text{ N}$$

$$E = 300 \text{ GPa}$$

$$\nu = 0.22$$

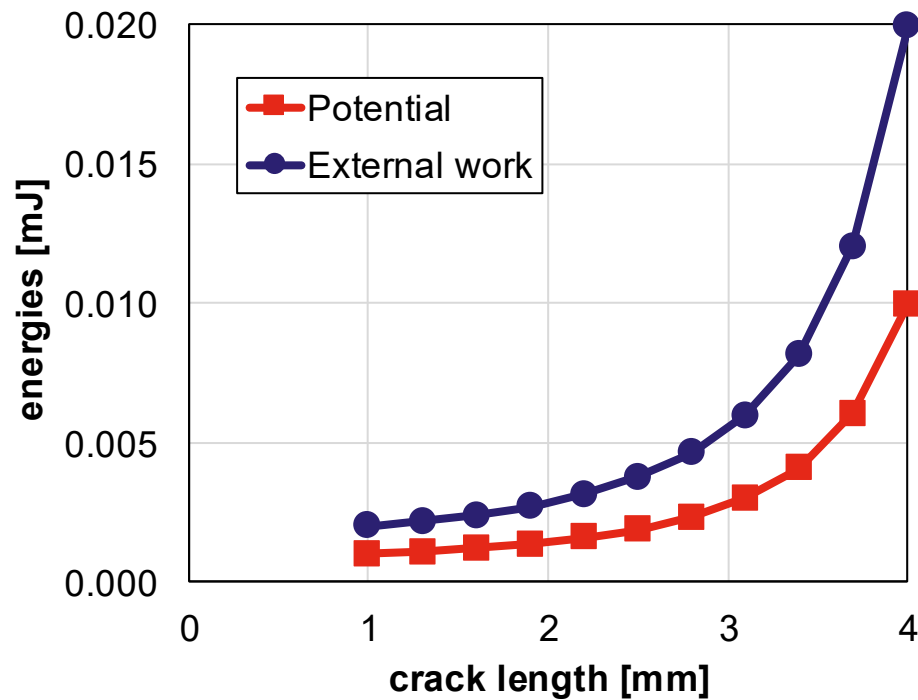


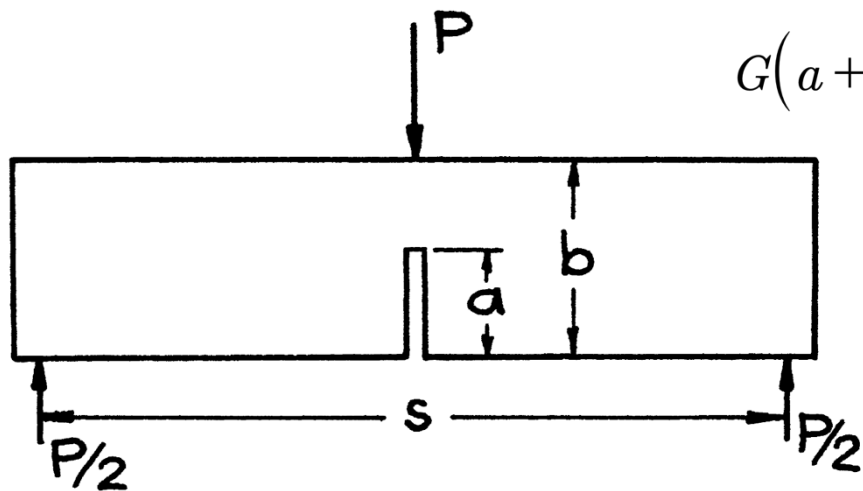


$$\sigma = \frac{6M}{b^2}$$

$$M = \frac{Ps}{4}$$

$$K_I = \sigma \sqrt{\pi a} F\left(\frac{a}{b}\right)$$



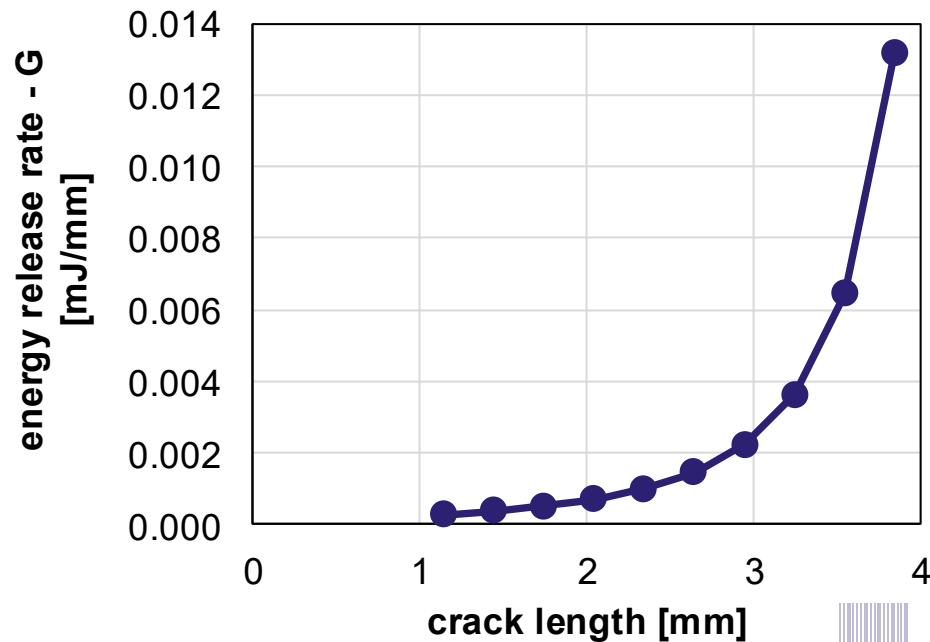


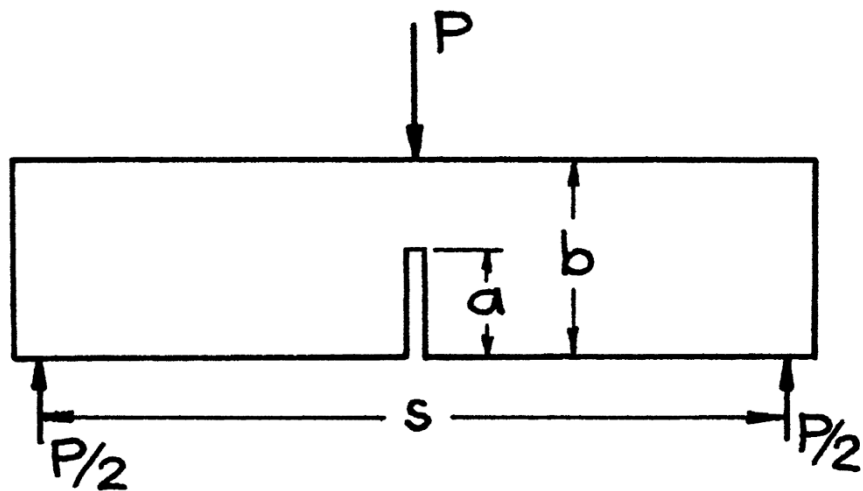
$$G\left(a + \frac{\Delta a}{2}\right) = -\frac{\Pi(a + \Delta a, P) - \Pi(a, P) - \Delta W_{ext}(P)}{\Delta a}$$

$$\sigma = \frac{6M}{b^2}$$

$$M = \frac{Ps}{4}$$

$$K_I = \sigma \sqrt{\pi a} F\left(\frac{a}{b}\right)$$



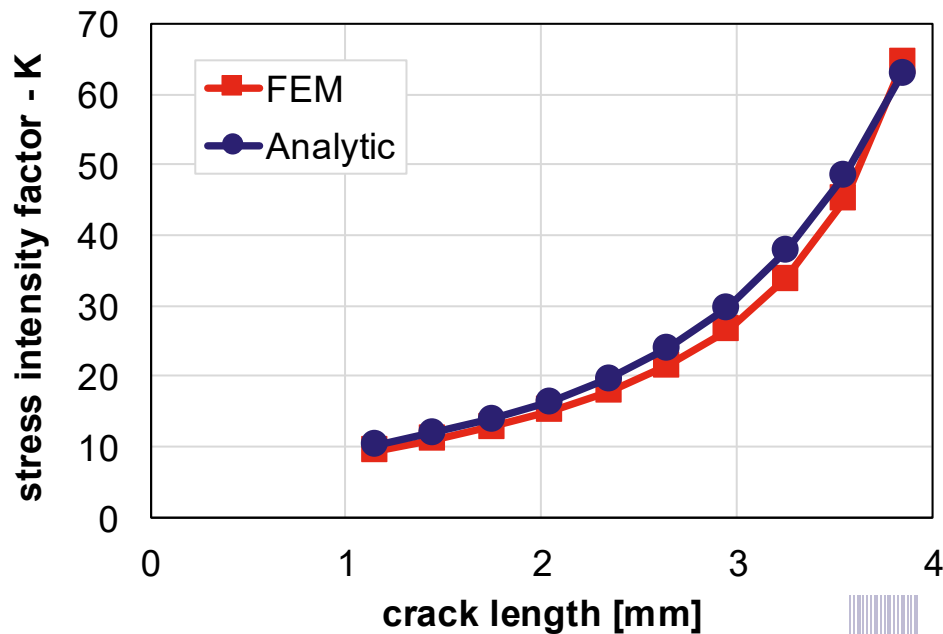


$$\sigma = \frac{6M}{b^2}$$

$$M = \frac{Ps}{4}$$

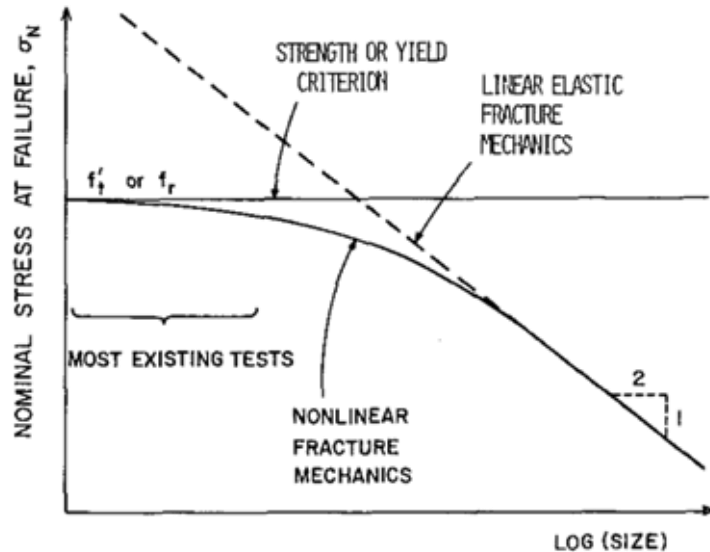
$$K_I = \sigma \sqrt{\pi a} F\left(\frac{a}{b}\right)$$

$$K = \sqrt{\frac{GE}{(1-\nu^2)}}$$



Motivation

Length scale in failure



(Bažant et al., 1984)

- concrete
- composites
- rocks
- ceramics
- silica glass
- wood
- SiC
- sea ice
- alumina
- etc...

Shah & Swartz (1987)

Bažant et al. (1996)

Bažant et al. (2004)

Usami et al. (1986)

Luo et al. (2016)

Simon (2009)

Bažant et al. (1990)

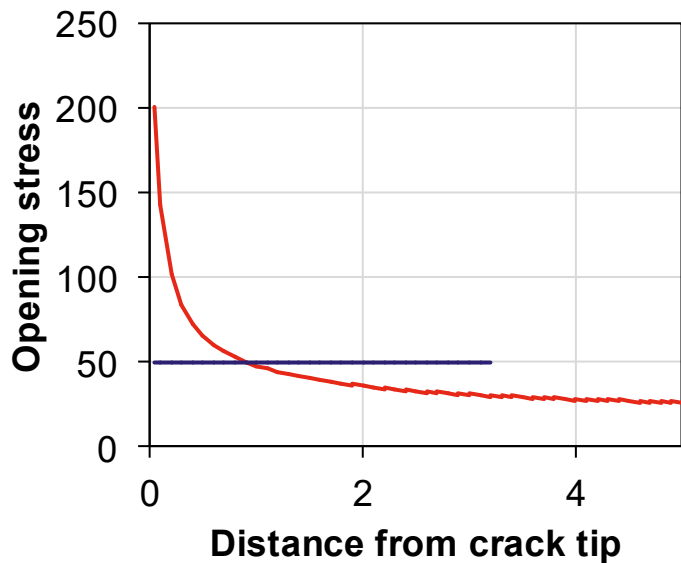
Dempsey et al. (1995)

Leguillon et al. (2018)



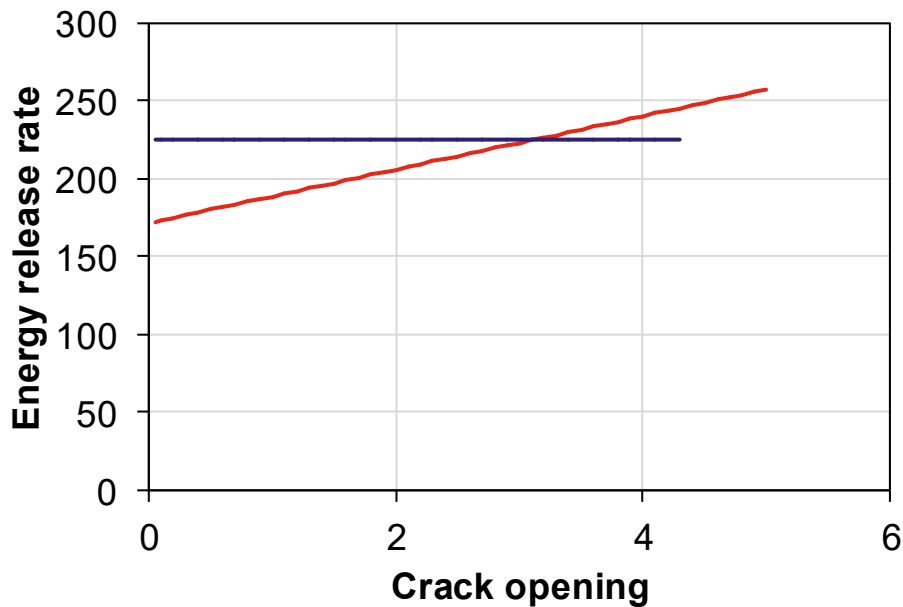
Coupled criterion

Tension



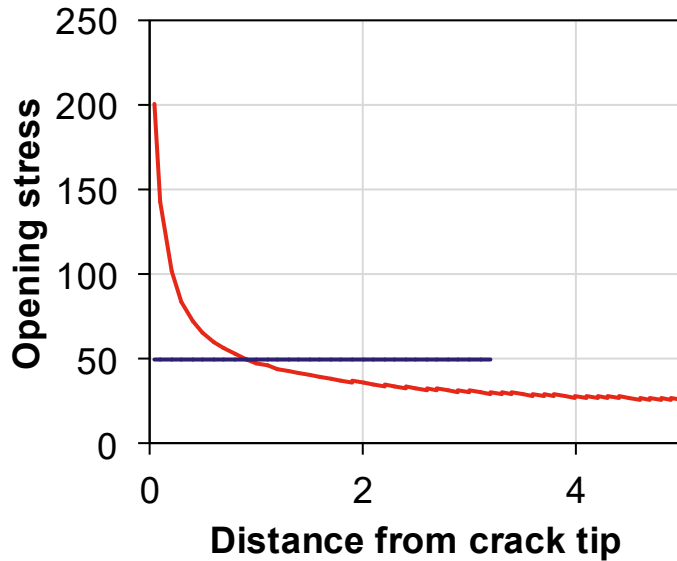
$$\sigma_y(r, \theta = 0) = \sigma_y^0 \frac{r+a}{\sqrt{r(r+2a)}} \geq \sigma_c$$

$$\frac{\partial \Pi}{\partial a} = G = \frac{\pi a}{2\mu} (1 - \nu) (\sigma_y^0)^2 \geq g_c$$



Coupled criterion

Tension



$$\frac{\Delta \Pi}{\Delta a} = G_{inc}(a + \Delta a) \geq g_c$$

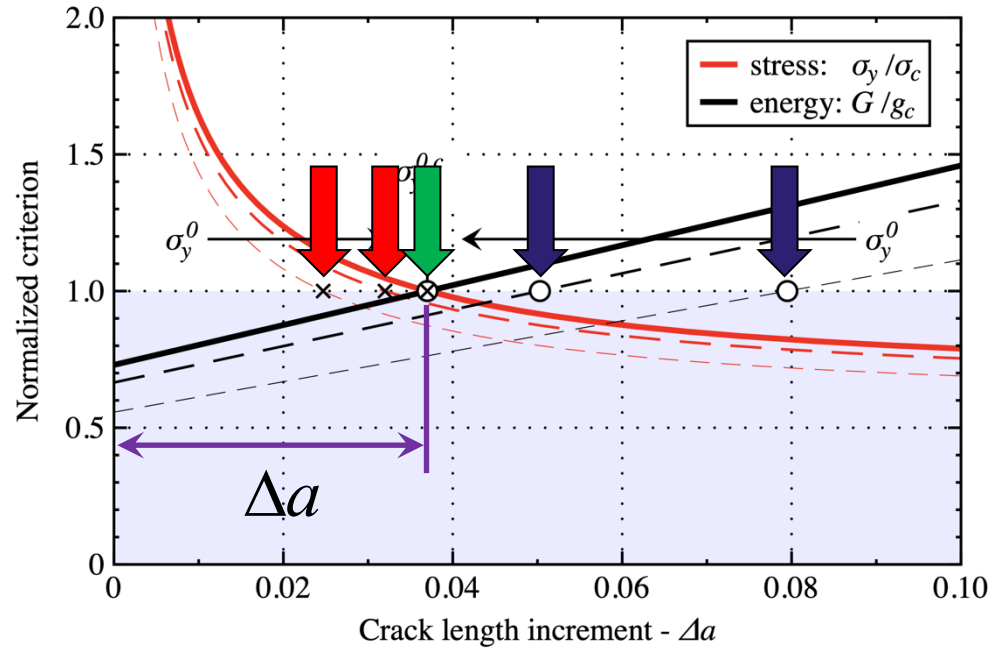
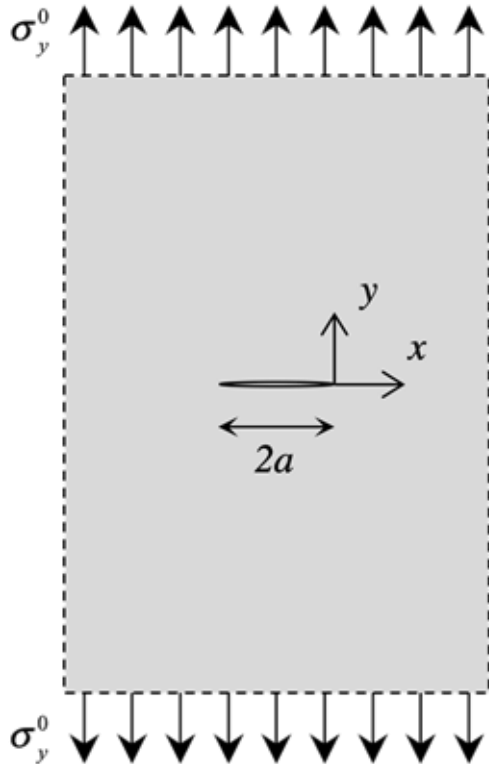
$$\lim_{\Delta a \rightarrow 0} \frac{\Delta \Pi}{\Delta a} = \frac{\partial \Pi}{\partial a}$$

$$G_{inc} = \frac{1}{\Delta a} \int_0^{\Delta a} G da$$

$$\sigma_y(r, \theta = 0) = \sigma_y^0 \frac{r + a}{\sqrt{r(r + 2a)}} \geq \sigma_c$$

Coupled criterion

Tension



Leguillon (2002)

$$G_{inc}(a + \Delta a) \geq g_c$$

Griffith

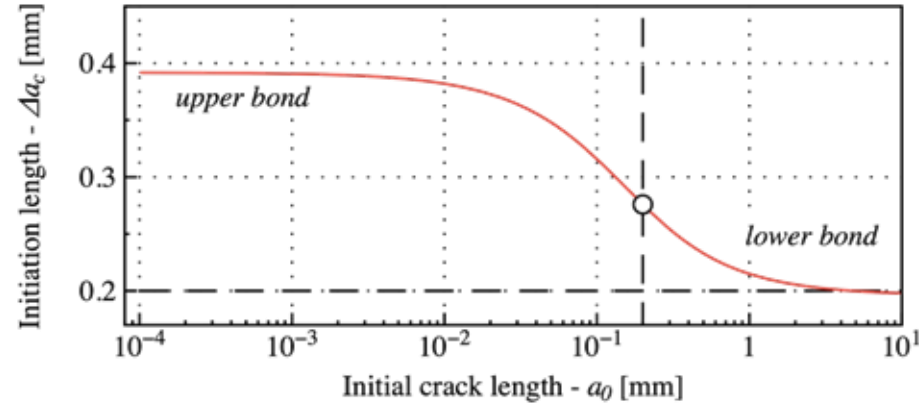
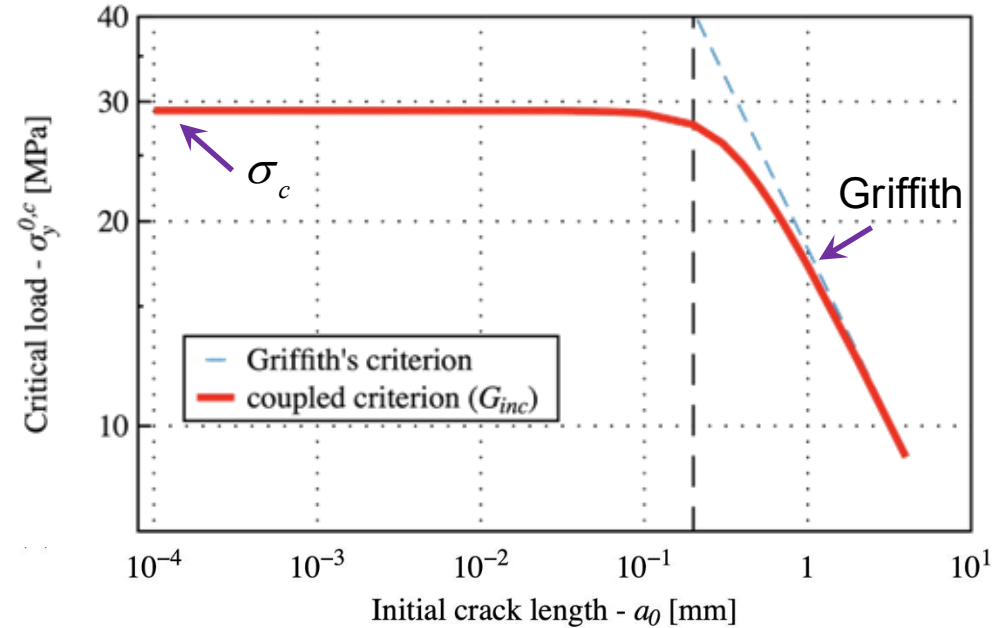
Stress

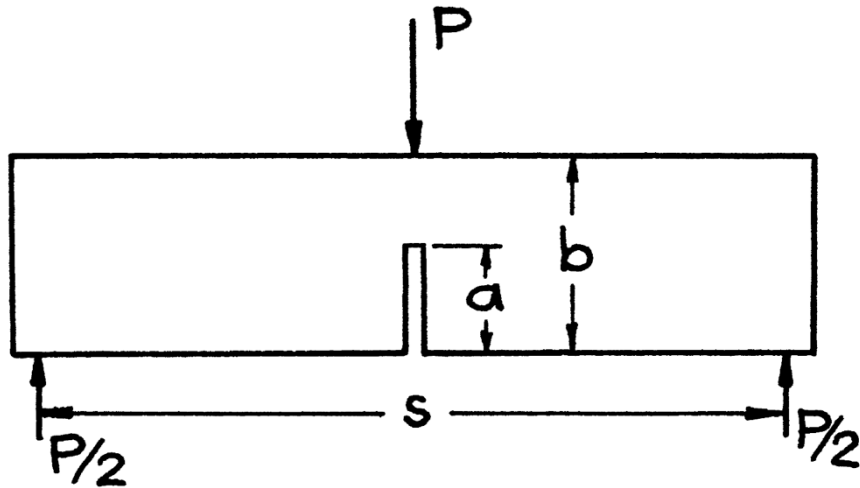
$$\sigma_1(a + \Delta a) \geq \sigma_c$$

Coupled **stress** and **energy** criteria

Tension

Size effect





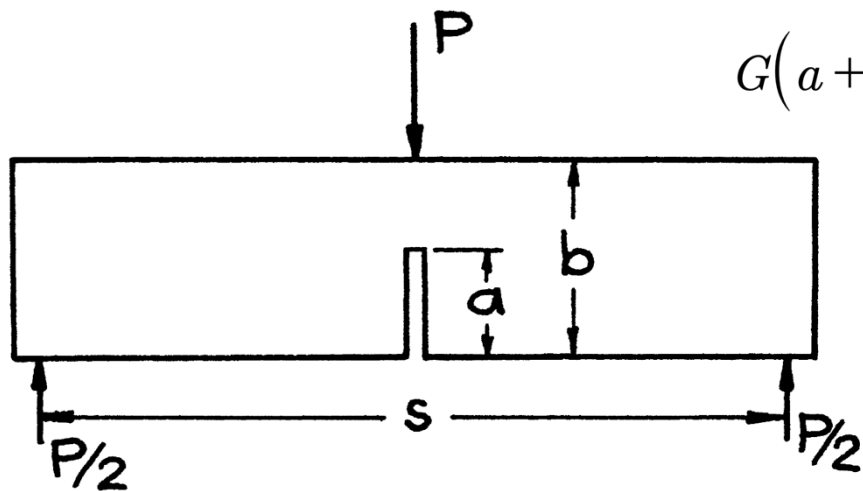
$$\sigma = \frac{6M}{b^2}$$

$$M = \frac{Ps}{4}$$

$$K_I = \sigma \sqrt{\pi a} F\left(\frac{a}{b}\right)$$

- 1) Analytical verification
 - a) Displacement field
 - b) Energy release rate
 - c) Effect of initial crack-tip roundness
- 2) Coupled criterion
 - a) Incremental energy release rate
 - b) Stress distribution
 - c) Criterion verification

Project

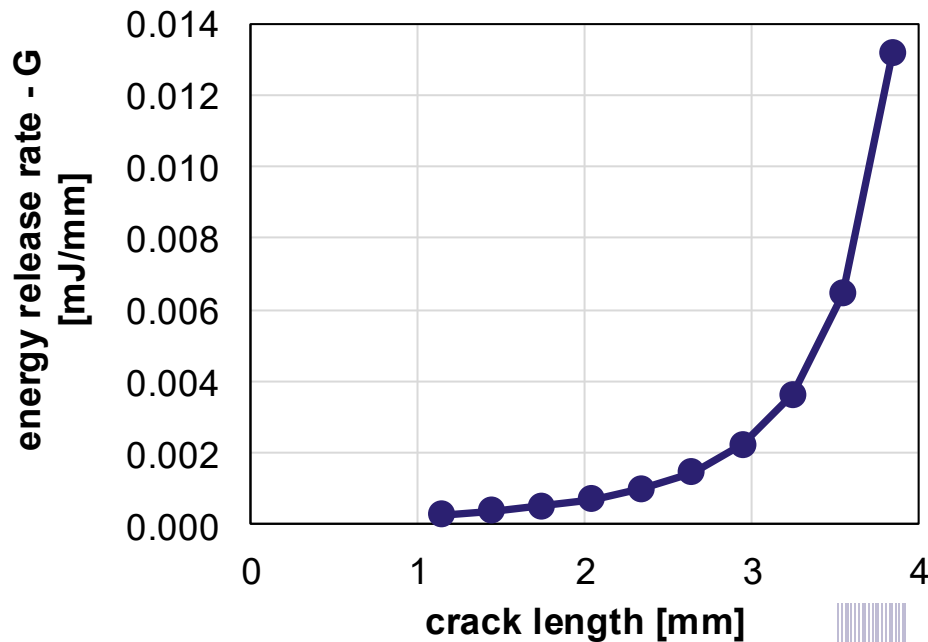


$$\sigma = \frac{6M}{b^2}$$

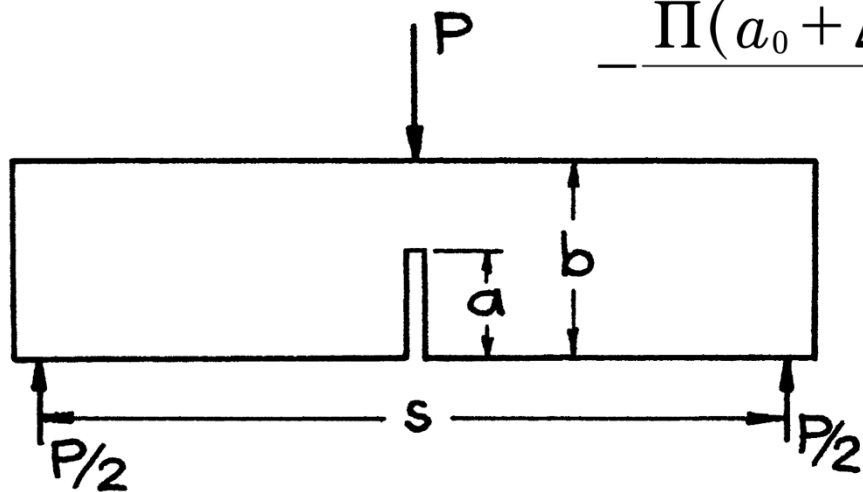
$$M = \frac{Ps}{4}$$

$$K_I = \sigma \sqrt{\pi a} F\left(\frac{a}{b}\right)$$

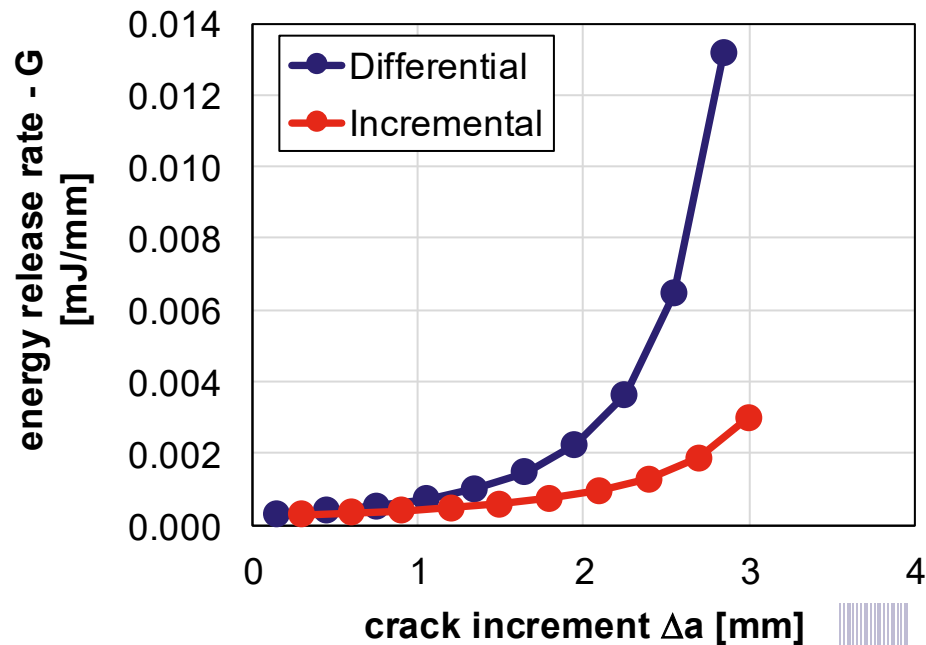
$$G\left(a + \frac{\Delta a}{2}\right) = -\frac{\Pi(a + \Delta a, P) - \Pi(a, P) - \Delta W_{ext}(P)}{\Delta a}$$



Project

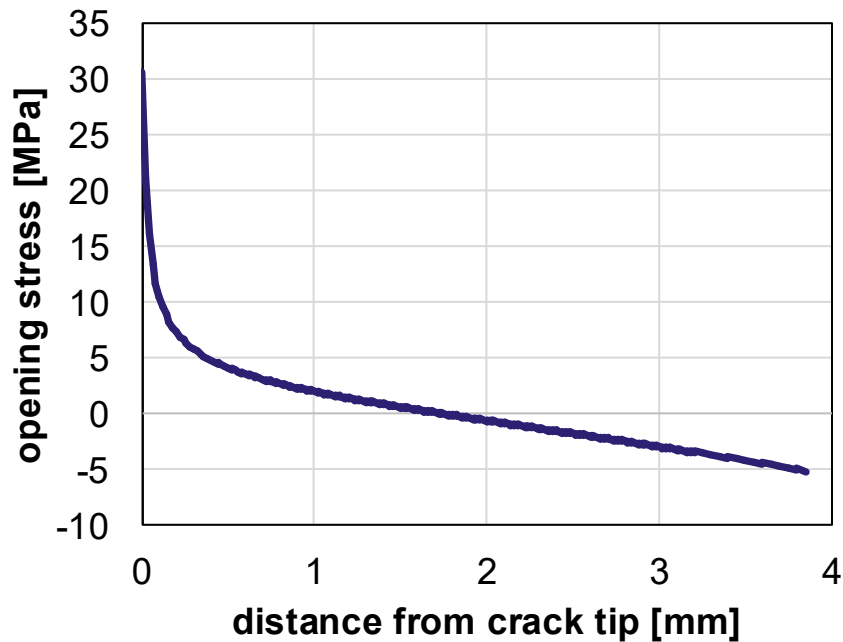


$$-\frac{\Pi(a_0 + \Delta a) - \Pi(a_0) - \Delta W_{ext}}{\Delta a} = G_{inc}(a_0 + \Delta a)$$

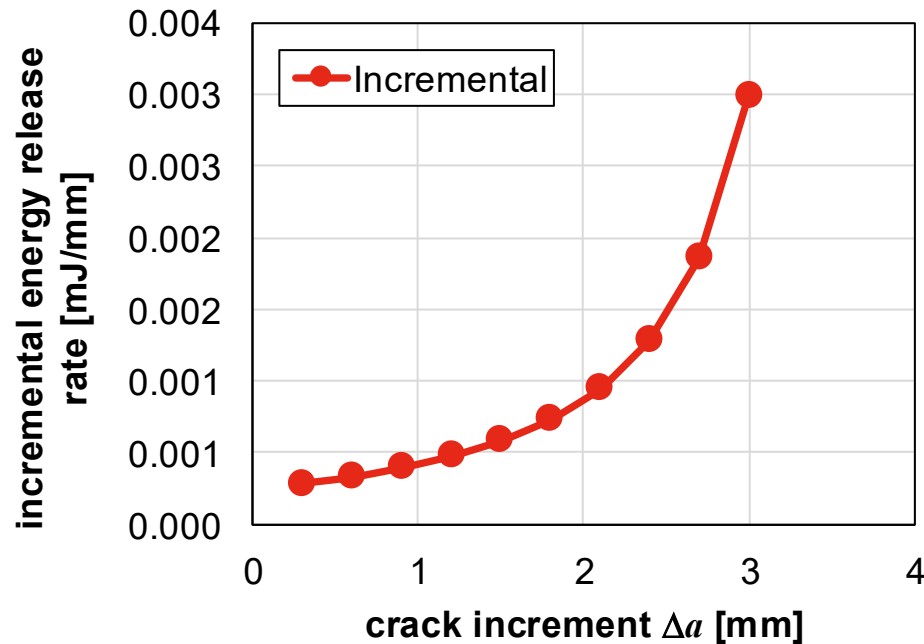


Project

$$\sigma_x \sim P$$



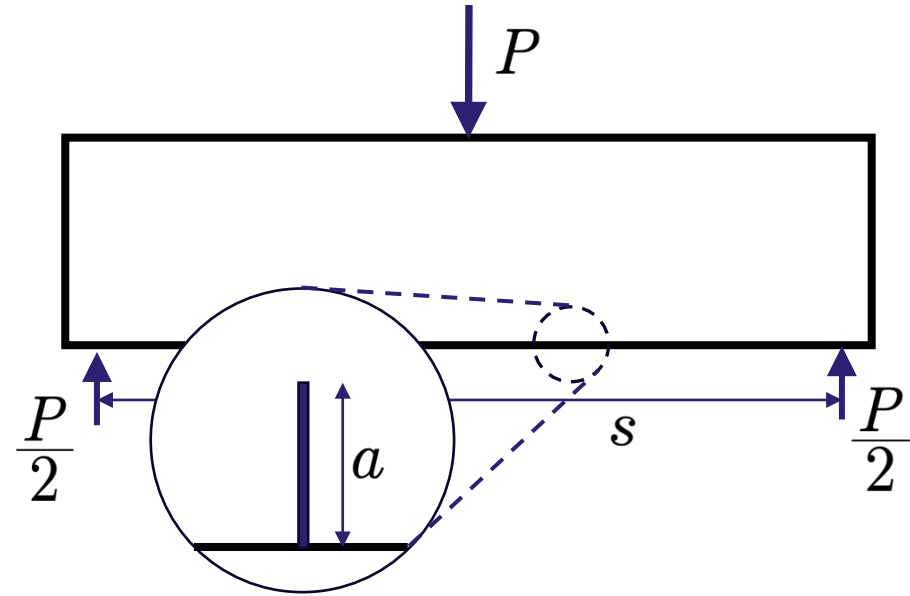
$$G_{inc} \sim P^2$$



$$(P_c, \Delta a_c) = \min \left\{ \frac{G_{inc}(a_0 + \Delta a)}{g_c}, \frac{\sigma_x(a_0 + \Delta a)}{\sigma_c} \right\} \geq 1$$

Coupled criterion

Matched asymptotic extension



$$K_I = 1.1215\sigma\sqrt{\pi a}$$

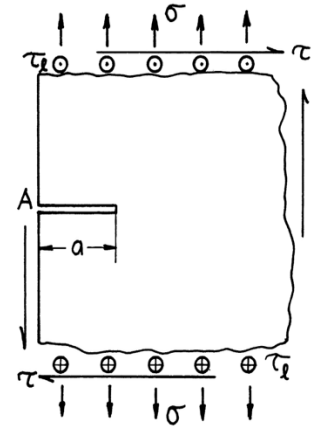
$$\sigma = \frac{6M}{b^2}$$

$$G = K_I^2 \frac{1-\nu^2}{E}$$

$$G_{inc} = \frac{1}{\Delta a} \int_{a_0}^{a_0+\Delta a} G(a) da$$

$$G_{inc} = \frac{1.1215^2 \sigma^2 \pi (2a_0 + \Delta a) (1-\nu^2)}{2E}$$

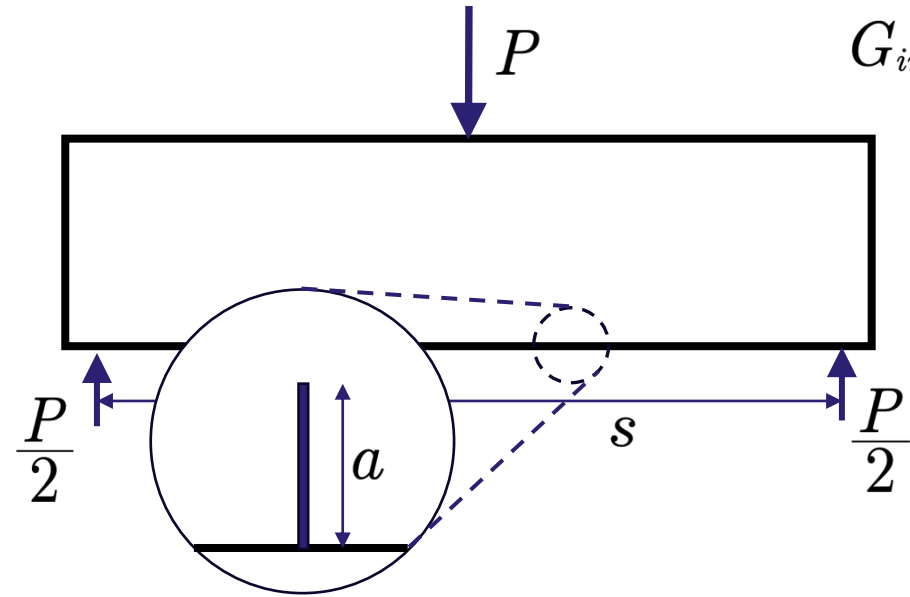
Energy release rate



Tada et al. (1973)

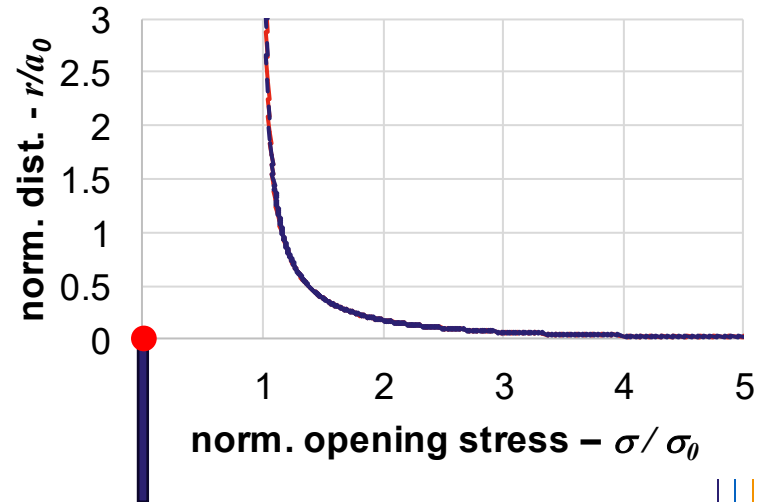
Coupled criterion

Matched asymptotic extension



$$G_{inc} = \frac{1.1215^2 \sigma_0^2 \pi (2a_0 + \Delta a)(1 - \nu^2)}{2E}$$

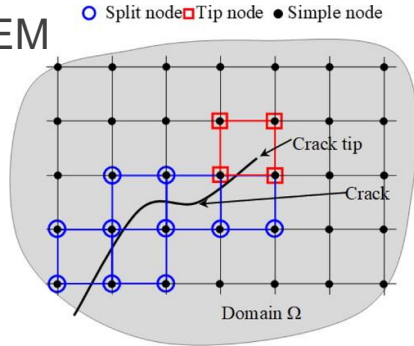
Stress



Modelling techniques

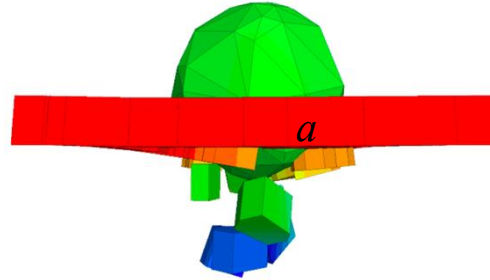
Discrete models

XFEM



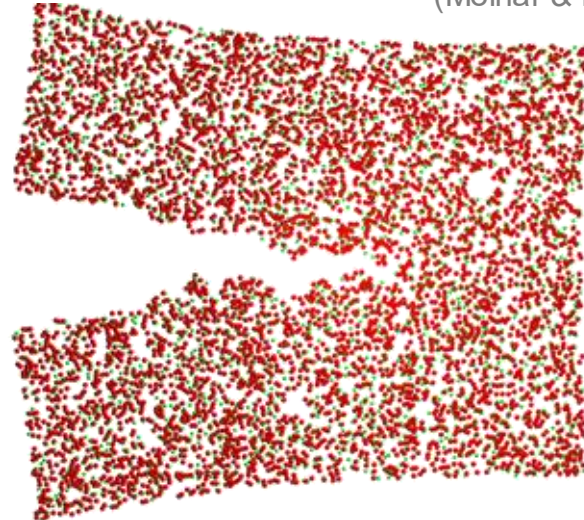
(Moës et al., 1999)
(Kired et al., 2024)

Discrete element models

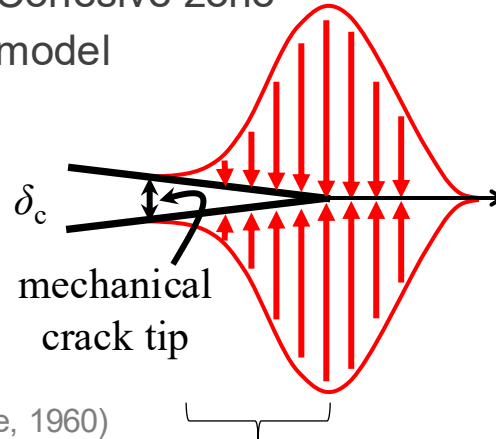


Molecular dynamics

(Molnar & Barthel, 2026)



Cohesive zone model



(Dugdale, 1960)
(Barenblatt, 1962)

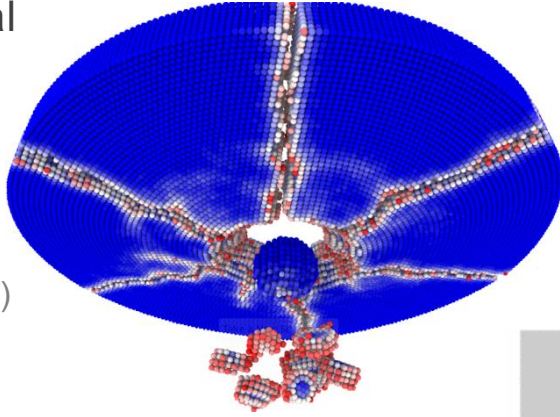
Modelling techniques

Diffuse models

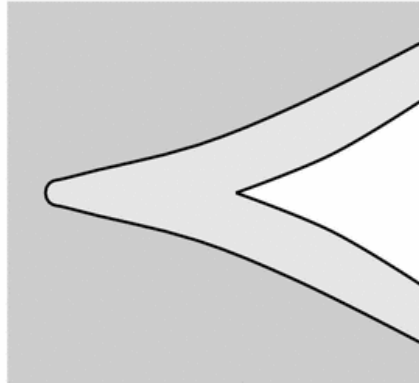
Non-local

PeriDyn
(Silling, 2000)

SPH

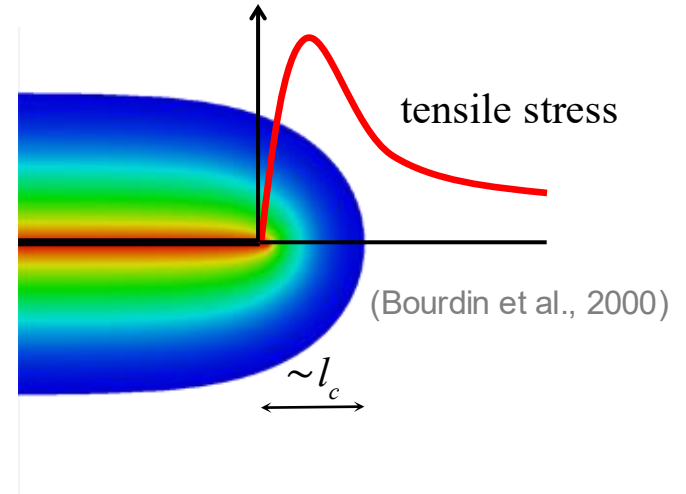


Thick level set



(Moës et al., 2011)

Phase-field



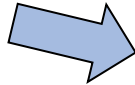
Phase-field model

Phase-field approach



liquid–gas interfaces
with **density**
functions

van der Waals (~1900)

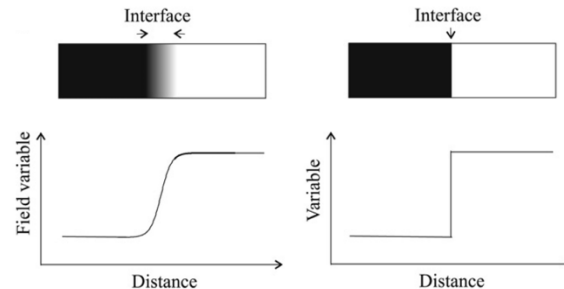


superconductivity

(Ginzburg & Landau, 1950)

diffuse interfaces

(Cahn & Hilliard, 1958)



L. M. Kachanov (1958)

continuum damage
mechanics

$$\psi = \frac{1}{2} (1 - d) \varepsilon_{ij} C_{ijkl} \varepsilon_{kl}$$



- **fracture mechanics**

(Bourdin et al., 2000)

- **solidification**

- **image processing** (Fix, 1983)
(Langer, 1986)



Phase-field model

Phase-field approach

1. Brittle fracture $-\frac{\partial \psi}{\partial a} = \frac{\partial S}{\partial a} = g_c$ (Griffith, 1920)



(Bourdin et al., 2000)

2. Minimization problem

$$\Pi(\mathbf{u}, \Gamma) = \int_{\Omega} \psi(\varepsilon(\mathbf{u})) d\Omega + g_c \int_{\Gamma} d\Gamma$$

(Mumford & Shah, 1989)
(Francfort & Marigo, 1998)

3. Crack energy density

$$\Pi(\mathbf{u}, d) = \int_{\Omega} g(d) \psi_0(\varepsilon(\mathbf{u})) d\Omega + \int_{\Omega} \frac{g_c}{c_{\omega} l_c} \left(\omega(d) + l_c^2 |\nabla d|^2 \right) d\Omega$$

crack energy density

$$l_c \rightarrow 0 \quad \Gamma \text{ converges}$$

$$\dot{d} \geq 0$$

$$c_{\omega} = 4 \int_0^1 \sqrt{\omega(s)} ds$$

(Ambrosio & Tortorelli, 1990)

(Bourdin et al., 2000)

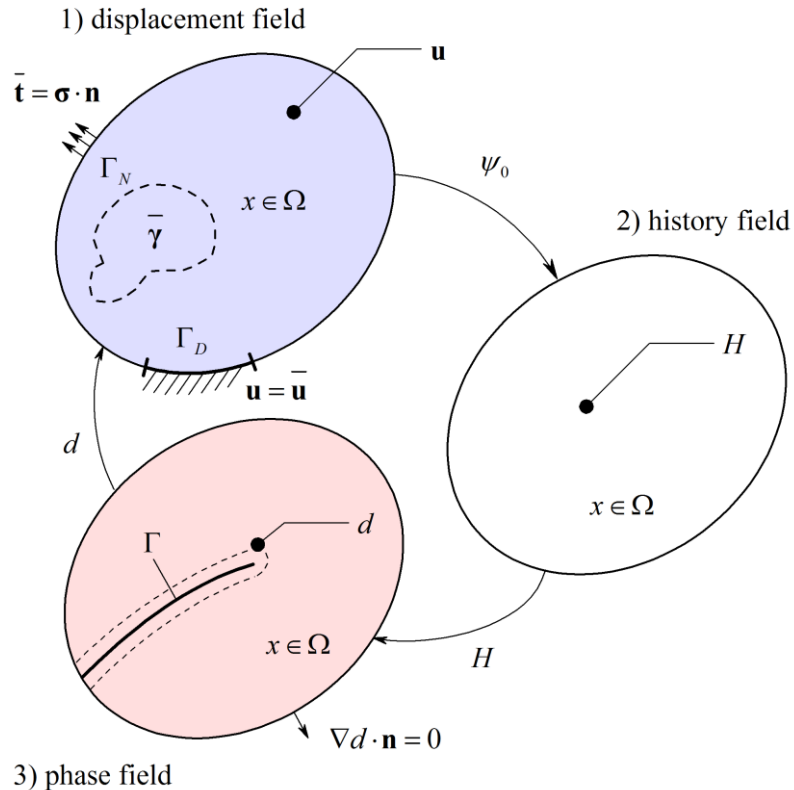
(Amor et al., 2009)

(Miehe et al., 2010)

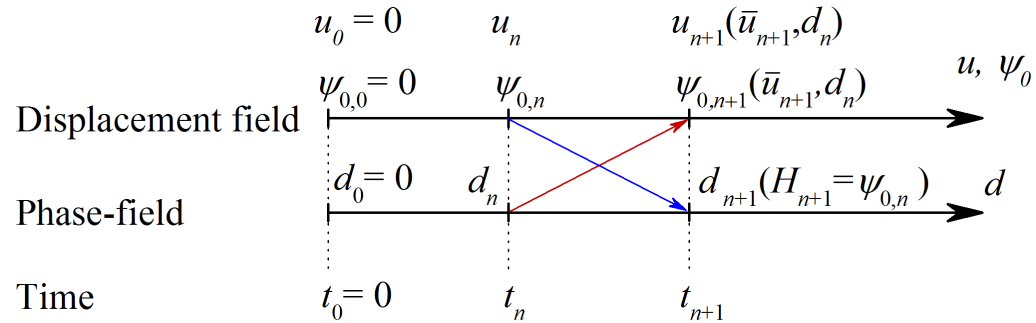


Phase-field model

Staggered solution



Staggered scheme



$$H = \begin{cases} \psi_0(\boldsymbol{\varepsilon}) & \text{if } \psi_0(\boldsymbol{\varepsilon}) > H_n \\ H_n & \text{otherwise,} \end{cases}$$

Phase-field model

Advantages

- **No need for ad-hoc criteria:**

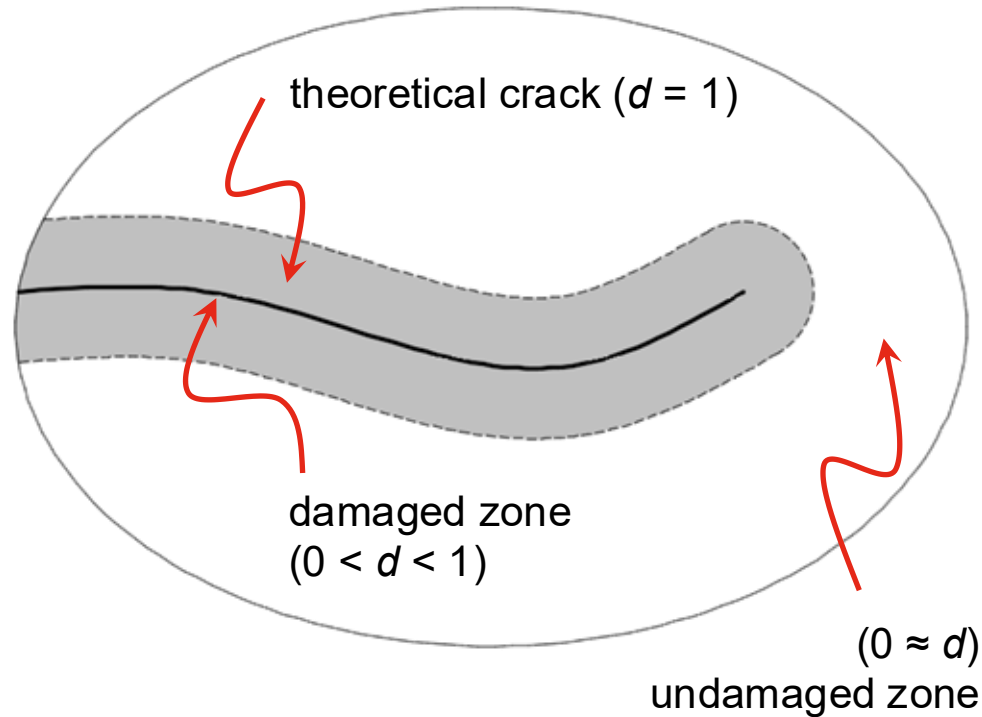
Initiation, propagation,
branching, coalescence

- 3D
- **Physically** sound

Disadvantages

- **Fine mesh**
- **Small** time **step**
- Numerical **experiment**

Solving **fracture mechanics** problems with
Partial Differential Equations (**PDEs**)



Phase-field model

Homogeneous solution

$$\Pi^d = \int_{\Omega} \left[\frac{g_c}{c_{\omega} l_c} [\omega(d) + l_c^2 |\nabla d|^2] + g(d) H \right] d\Omega$$

$$\nabla d = \mathbf{0} \quad \text{homogeneous}$$

$$\frac{\partial \Pi^d}{\partial d} = \frac{g_c}{c_{\omega} l_c} \frac{\partial \omega(d)}{\partial d} + \frac{\partial g(d)}{\partial d} H = 0$$

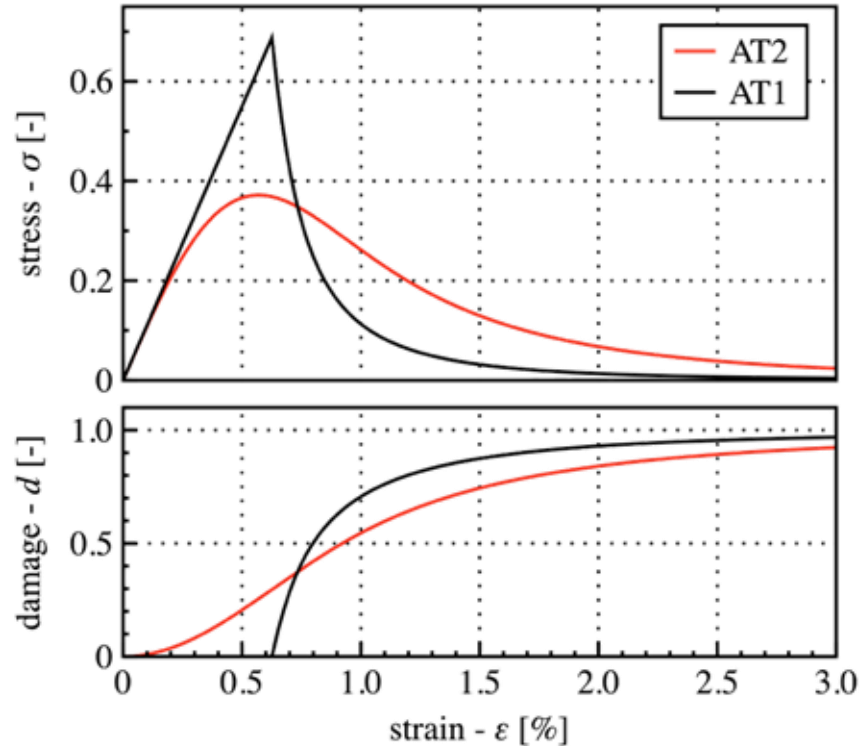
$$H = \frac{1}{2} \epsilon_y^2 C \quad g(d) = (1-d)^2 \longrightarrow \frac{\partial g(d)}{\partial d} = -2(1-d)$$

$$d = \frac{2H}{2H + \frac{g_c}{l_c}} \quad \text{AT2} \quad C = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \quad \text{if } \epsilon_x = \gamma_{xy} = 0$$

$$\begin{array}{l} \text{AT1} \left\{ \begin{array}{l} \omega(d) = d \\ c_{\omega} = 8/3 \end{array} \right. \\ \text{AT2} \left\{ \begin{array}{l} \omega(d) = d^2 \\ c_{\omega} = 2 \end{array} \right. \end{array}$$

Phase-field model

Homogeneous solution

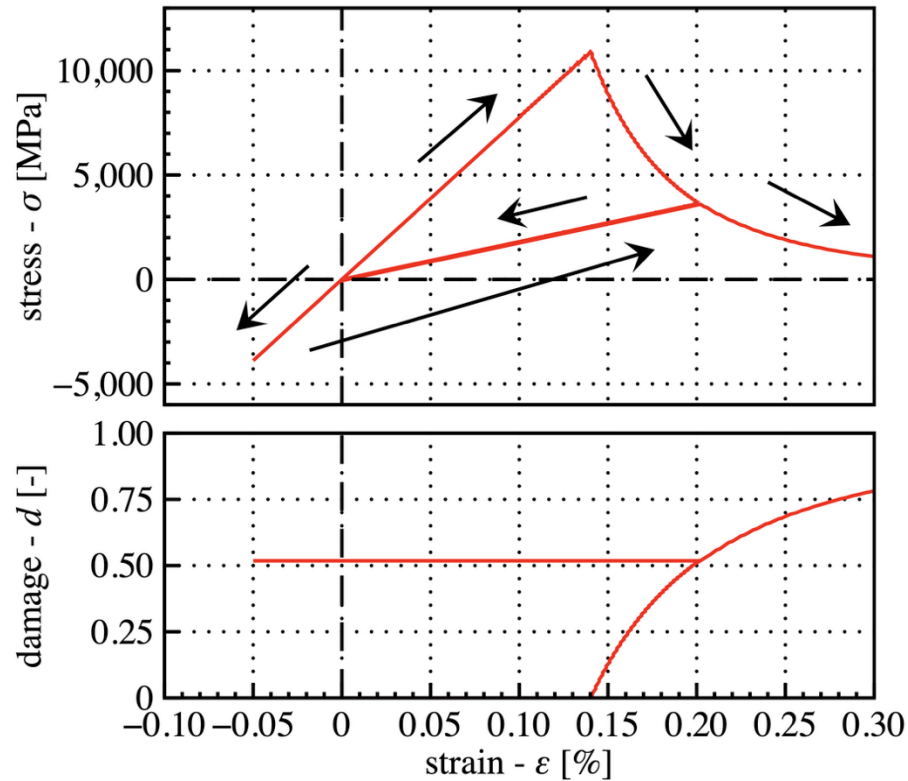


$$\begin{aligned} \text{AT1} & \left\{ \begin{array}{l} \omega(d) = d \\ c_\omega = 8/3 \end{array} \right. \\ \text{AT2} & \left\{ \begin{array}{l} \omega(d) = d^2 \\ c_\omega = 2 \end{array} \right. \end{aligned}$$

$$\frac{\partial \psi}{\partial \epsilon_y} = \sigma_y = (1 - d)^2 C$$

Phase-field model

Tension / compression



$$\psi(\boldsymbol{\varepsilon}, d) = g(d) \psi_0^+(\boldsymbol{\varepsilon}) + \psi_0^-(\boldsymbol{\varepsilon})$$

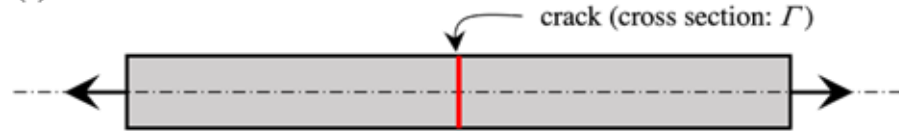
$$\psi_0^{+,-}(\boldsymbol{\varepsilon}) = \frac{\lambda}{2} \langle \text{tr}(\boldsymbol{\varepsilon}) \rangle_{+,-}^2 + \mu \sum_{i=1}^3 \langle \hat{\varepsilon}_i \rangle_{+,-}^2$$

$$\langle x \rangle_{+,-} = \frac{x \pm |x|}{2}$$

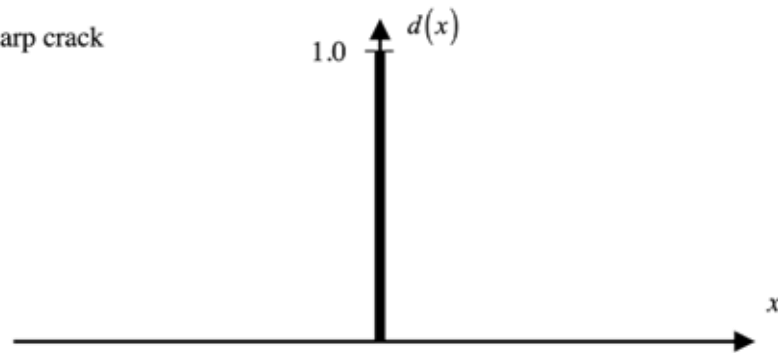
1D non-homogeneous solution

$$\begin{aligned} \delta \Pi^d = 0 \quad \forall \delta d &\rightarrow \frac{g_c}{l_c c_\omega} \left(\frac{\partial \omega}{\partial d} - l_c^2 \Delta d \right) = 0 && \text{in } \Omega \\ \nabla d \cdot \mathbf{n} &= 0 && \text{on } \Gamma \\ \dot{d} &\geq 0 && \text{in } \Omega \end{aligned}$$

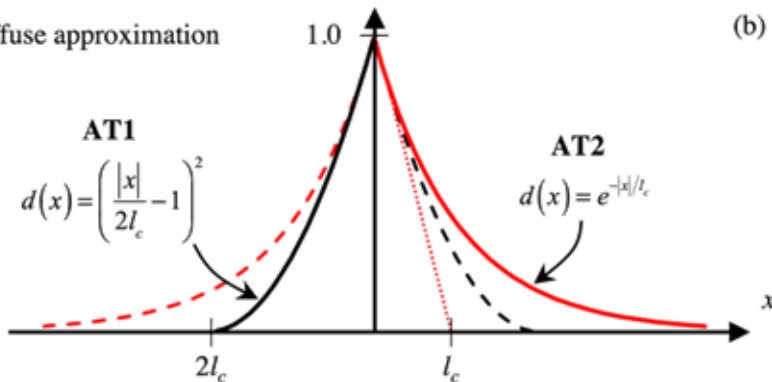
(a) cracker 1D bar



(b) sharp crack

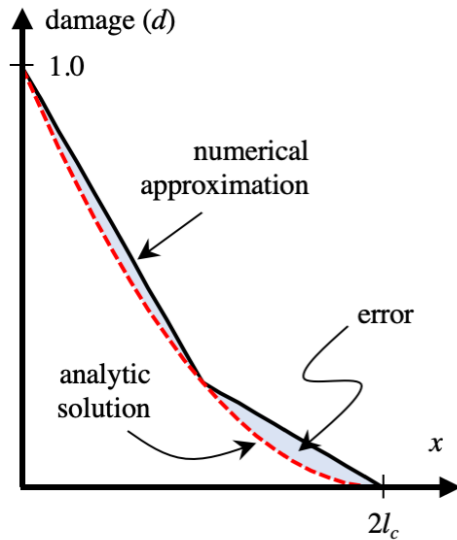


(c) diffuse approximation

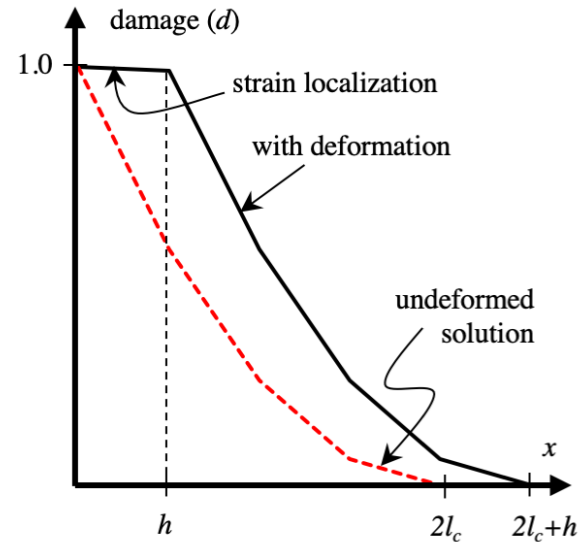


Different errors (spatial)

Approximation error

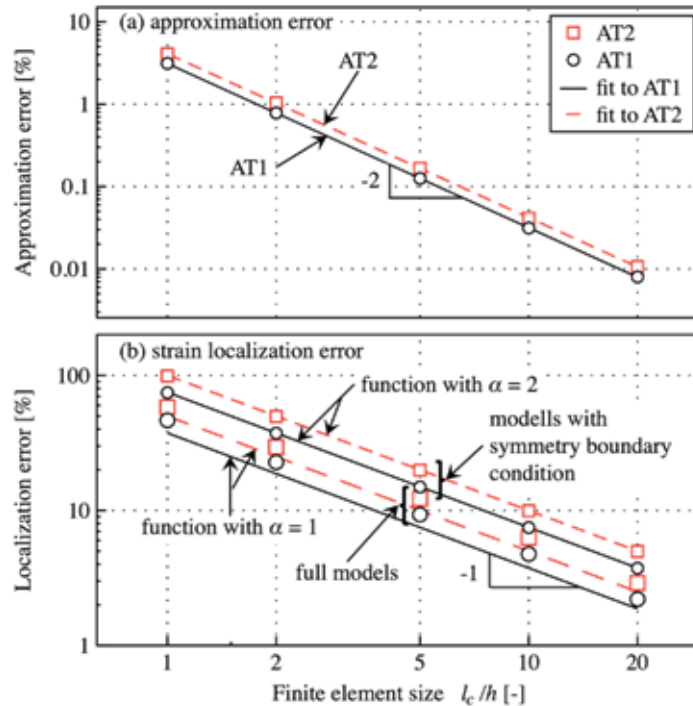


Localization error

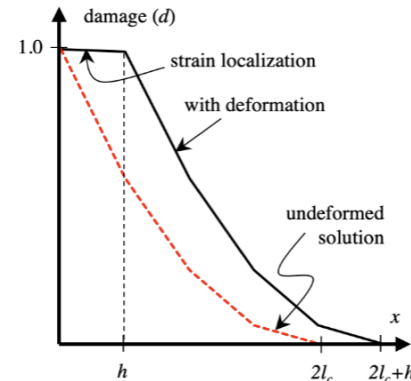


Different errors (spatial)

Approximation error



Localization error



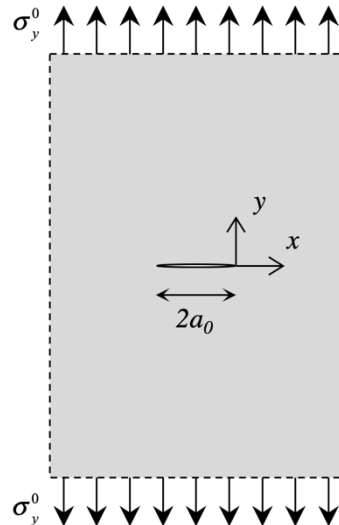
$$g_c = \frac{g_{mat}}{1 + \zeta}$$

$$\zeta = \alpha \frac{h}{c_\omega l_c}$$

Different errors (spatial)

Test correction

(a) Mode I: tensile opening

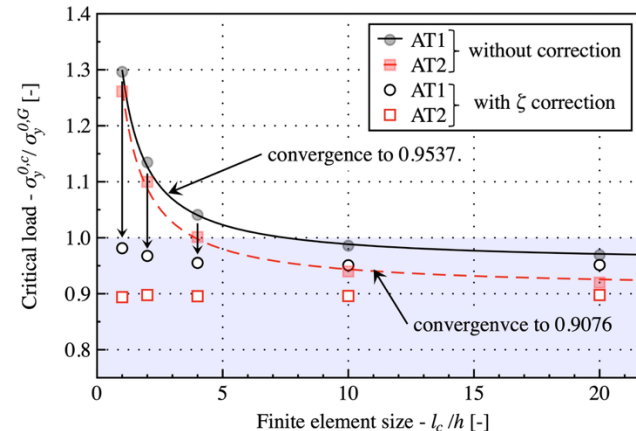
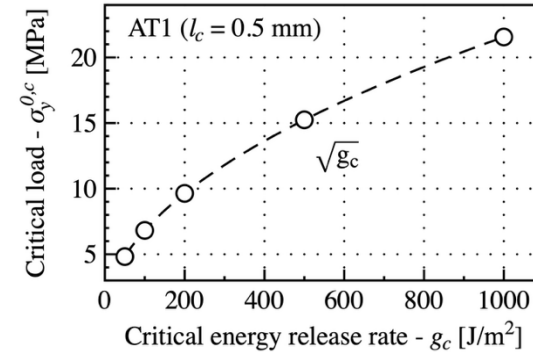


$$\sigma_y^{0,G} = \sqrt{\frac{8\mu g_{mat}}{(\kappa+1)a\pi}}$$

Griffith (1921,1924)

$$g_c = \frac{g_{mat}}{1+\zeta}$$

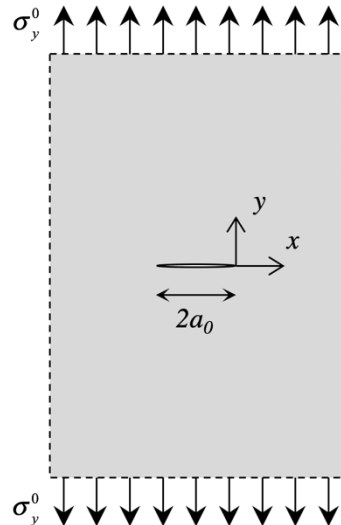
$$\zeta = \alpha \frac{h}{c_\omega l_c}$$



Different errors (spatial)

Test correction

(a) Mode I: tensile opening

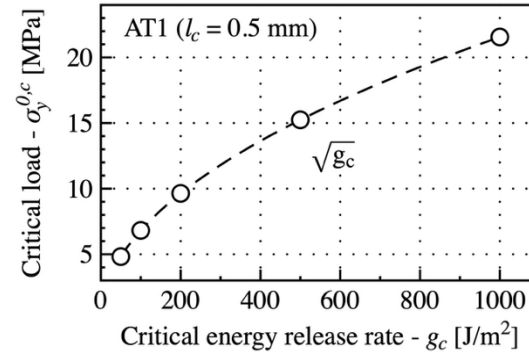


$$\sigma_y^{0,G} = \sqrt{\frac{8\mu g_{mat}}{(\kappa + 1)a\pi}}$$

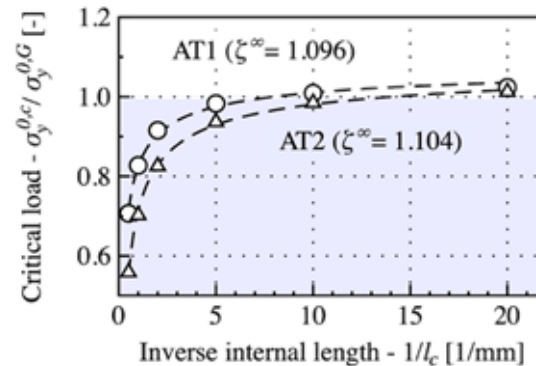
Griffith (1921,1924)

$$g_c = \frac{g_{mat}}{1 + \zeta}$$

$$\zeta = \alpha \frac{h}{c_\omega l_c}$$



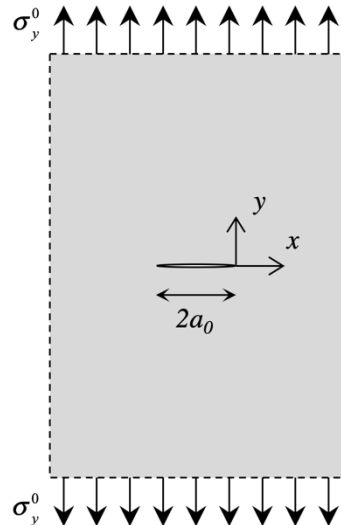
$l_c/h = 10$



Different errors (spatial)

Test correction

(a) Mode I: tensile opening

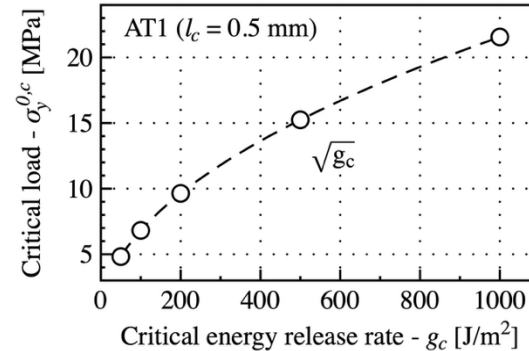


$$\sigma_y^{0,G} = \sqrt{\frac{8\mu g_{mat}}{(\kappa + 1)a\pi}}$$

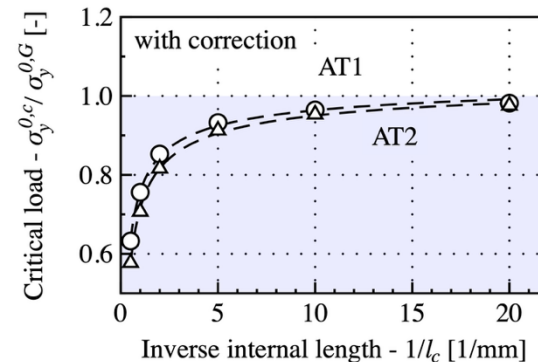
Griffith (1921,1924)

$$g_c = \frac{g_{mat}}{1 + \zeta}$$

$$\zeta = \alpha \frac{h}{c_\omega l_c}$$

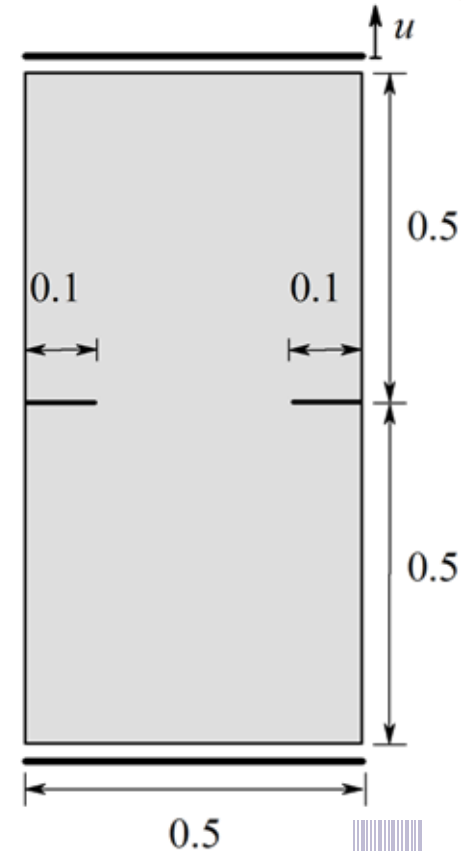
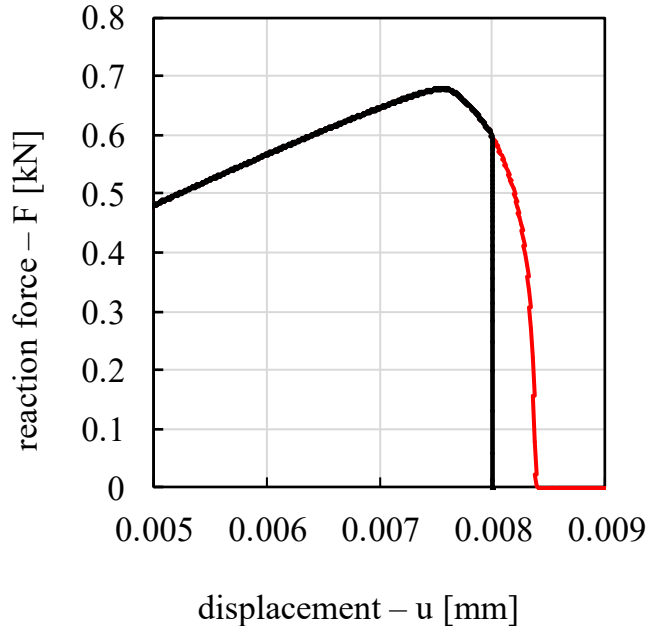
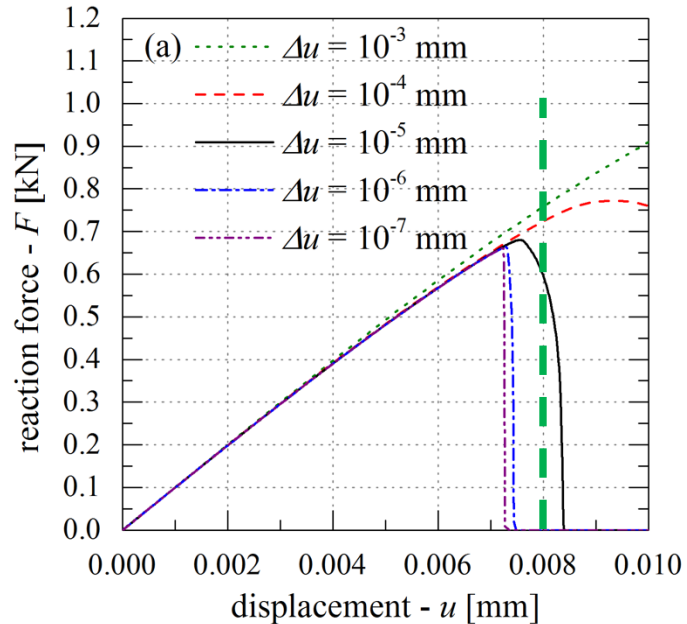


$l_c/h = 10$



Phase-field model

Different errors (temporal)

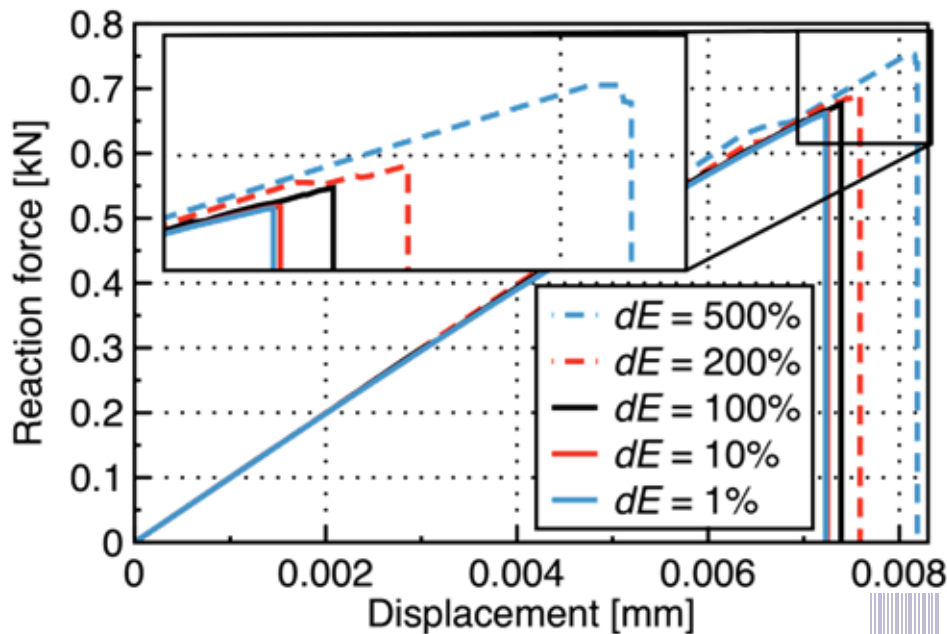
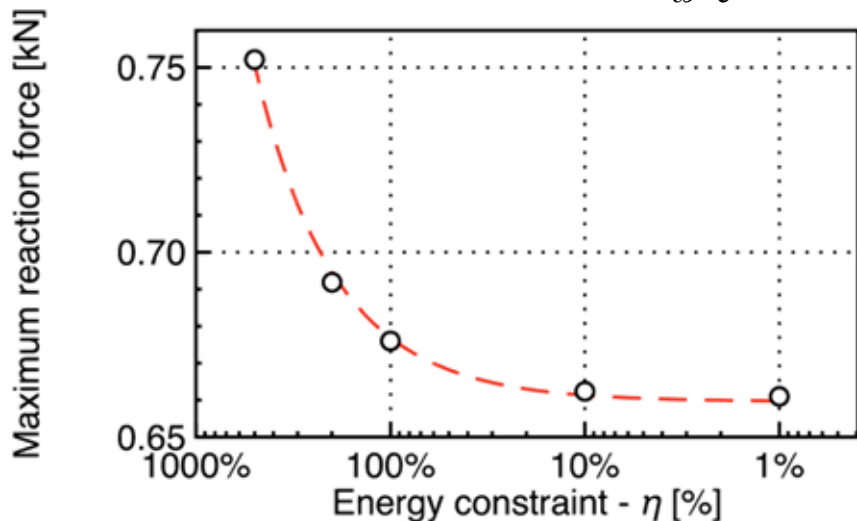


Molnár & Gravouil (2017)

Different errors (temporal)

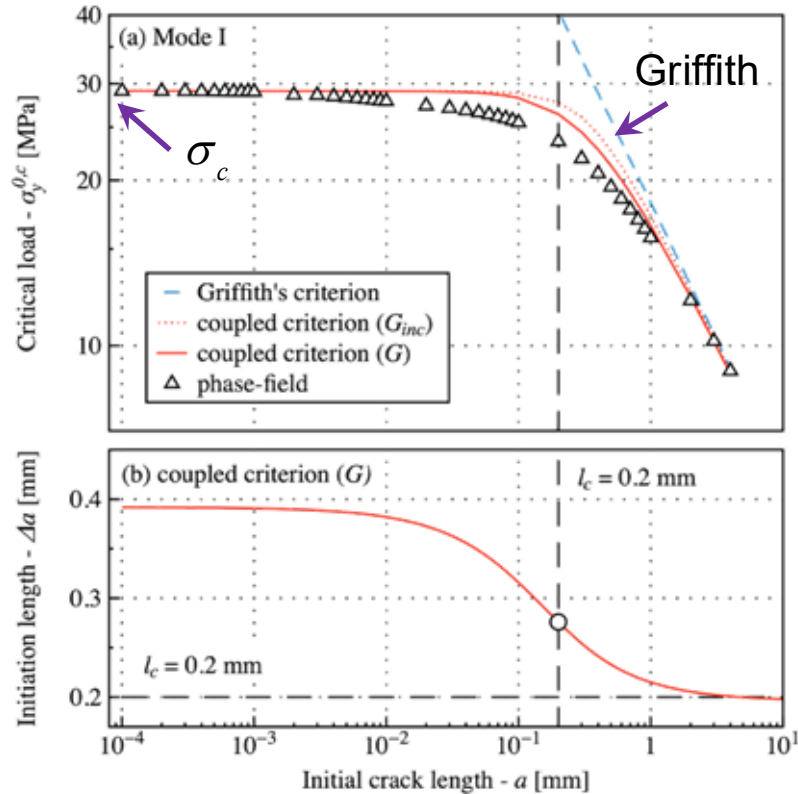
$$H = \begin{cases} \psi_0(\boldsymbol{\varepsilon}) & \text{if } \psi_0(\boldsymbol{\varepsilon}) > H_n \\ H_n & \text{otherwise} \end{cases} \quad \Rightarrow \quad d_{n+1} = \text{Arg} \left\{ \inf_d \int_{\Omega} \left[g_c \gamma(d, \nabla d) + (1-d)^2 H \right] dV \right\}$$

$$dH \leq dE^{\text{lim}} \quad \Rightarrow \quad dE^{\text{lim}} = \frac{g_c}{c_{\omega} l_c} \square \eta$$

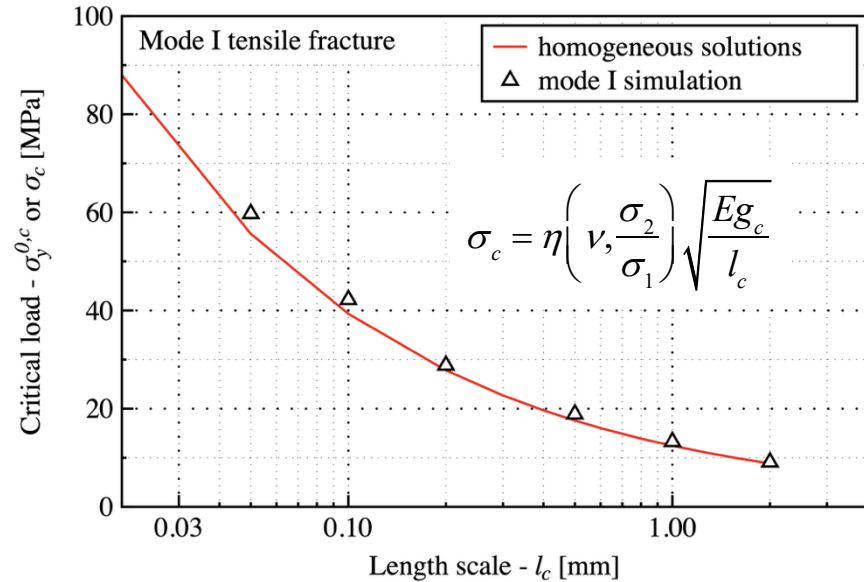


Comparison

Size effect

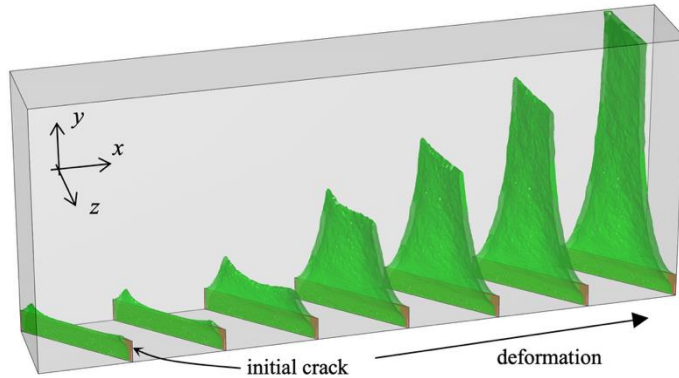


Correlation



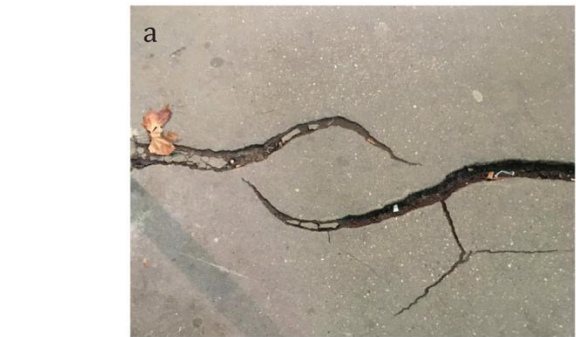
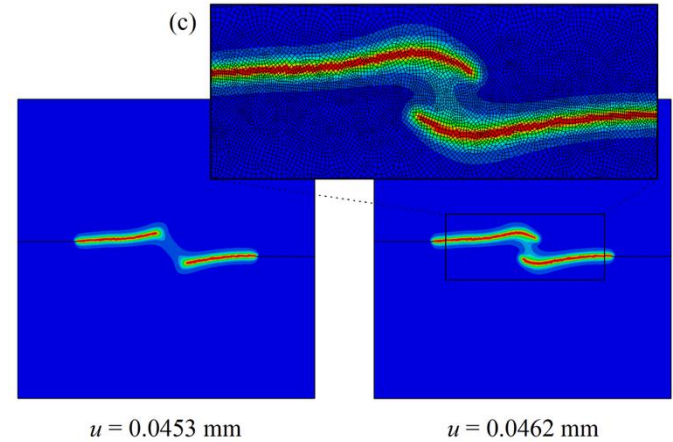
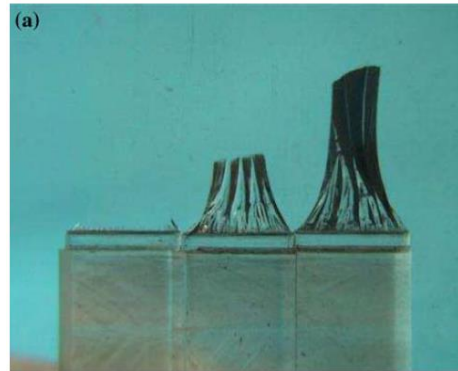
Phase-field model

Examples



Mode I+III

Lazarus et al. (2008)



Mode I+II

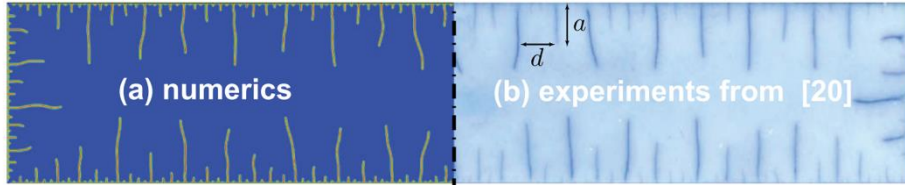


Phase-field model

Examples

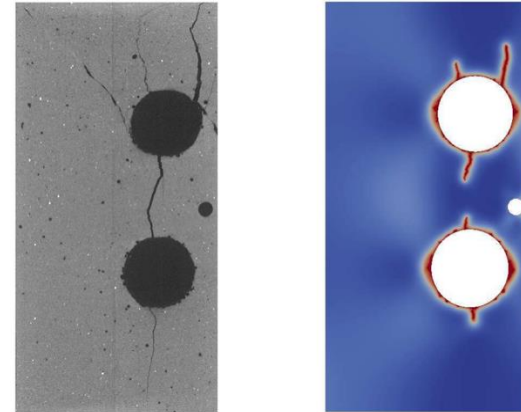
Thermal shock

Bourdin et al. (2014)



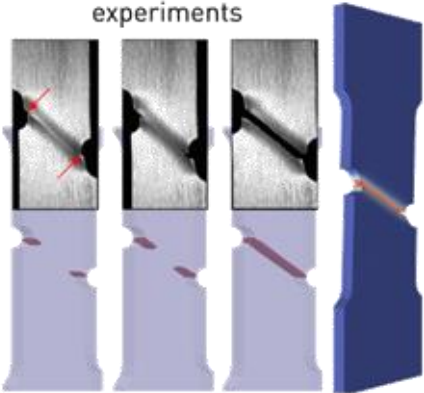
Heterogenous materials

Nguyen et al. (2016)



Ductile fracture

experiments

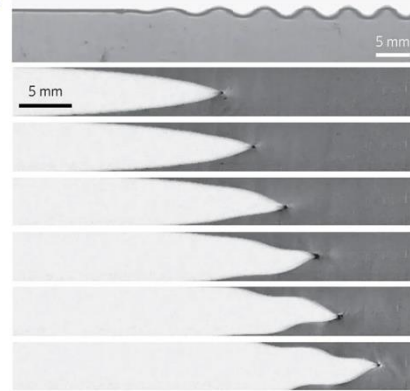
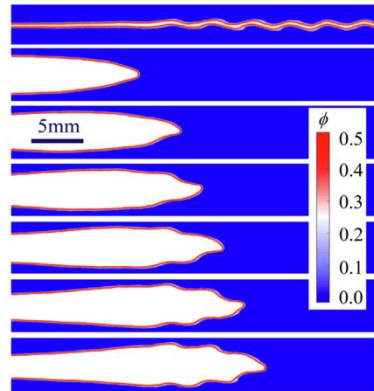


model predictions

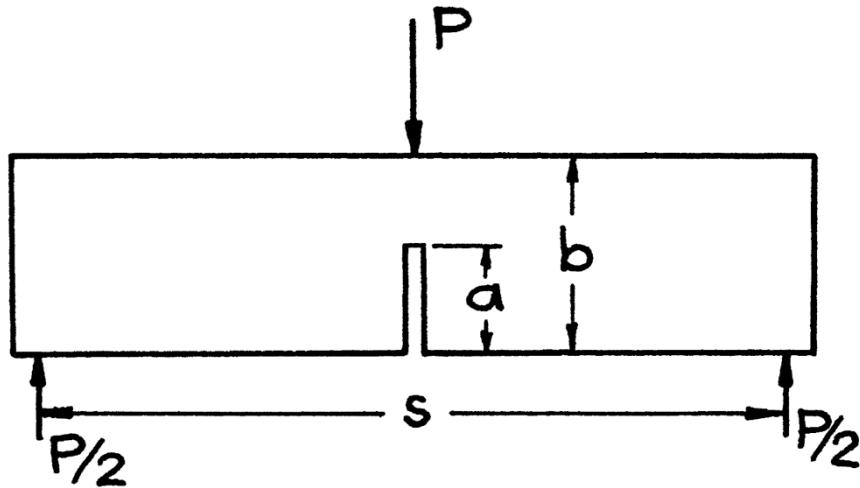
Ambati et al. (2015)

Hyperelastic fracture

(b)



Tian et al. (2020)



$$\sigma = \frac{6M}{b^2}$$

$$M = \frac{Ps}{4}$$

$$K_I = \sigma \sqrt{\pi a} F\left(\frac{a}{b}\right)$$

- 1) Analytical verification
 - a) Displacement field
 - b) Energy release rate
 - c) Effect of initial crack-tip roundness
- 2) Coupled criterion
 - a) Incremental energy release rate
 - b) Stress distribution
 - c) Criterion verification
- 3) Phase-field
 - a) Homogeneous solution
 - b) Fracture resistance

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