



Finite element method

Gergely Molnár



Definition

The **finite element method** is based on **discretizing** both the **geometry** and the **function space** using appropriate **basis functions**.

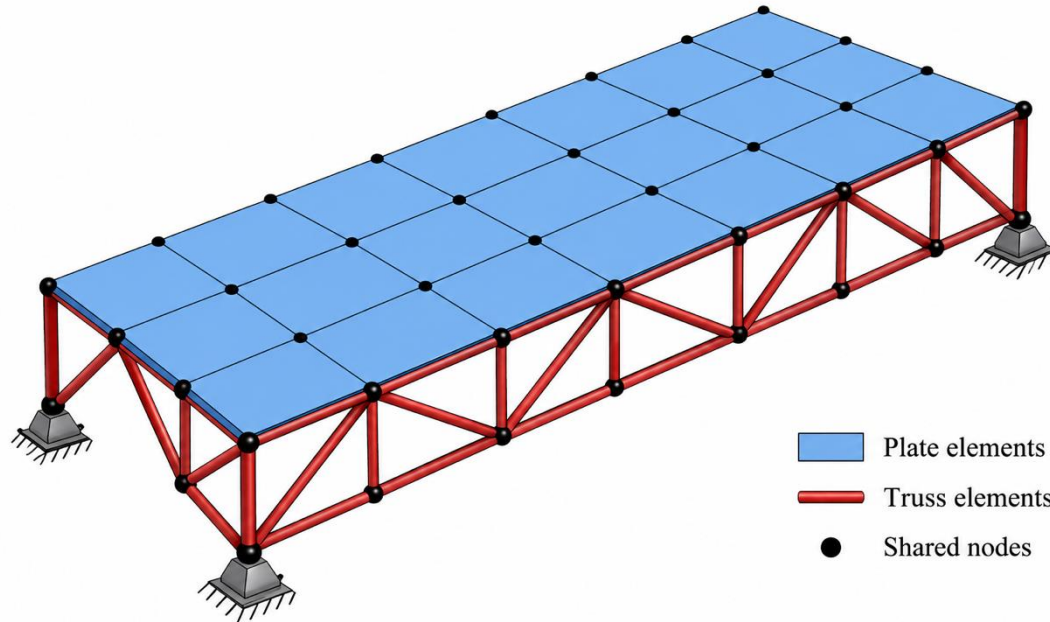
Steps

1. **Subdivision** of a whole domain into simpler parts
2. Discretization of the **function space**
3. Construction of the **matrices** associated with the **elements**
4. Assembly of the **system of equations** for the entire structure.
5. **Solution** of the system of equations
6. Calculation of the **secondary variables** (e.g. stresses)

Subdivision

Accurate representation of complex geometry

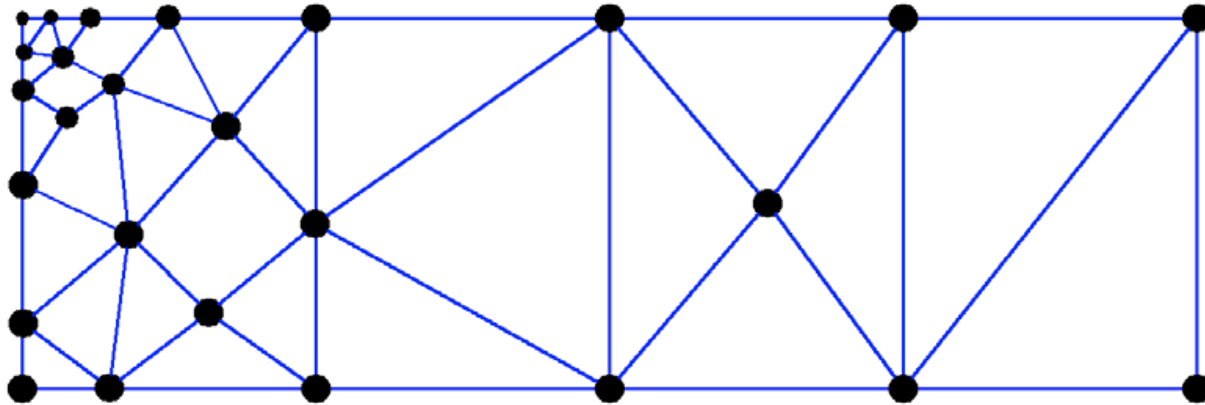
- What type of elements?



Subdivision

Accurate representation of complex geometry

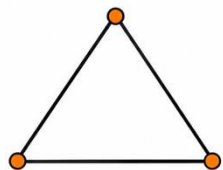
- What type of elements?
- How many element?
- Where to have nodes?



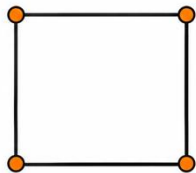
Function space

Choice of element type

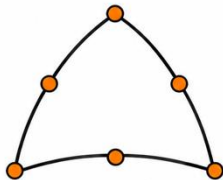
- Shape functions
- Number of integration points
- Shape of the element (shear locking)



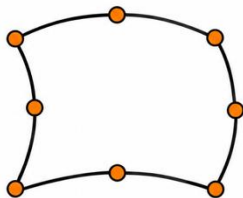
2-D 3-node
triangle



2-D 4-node
quadrilateral

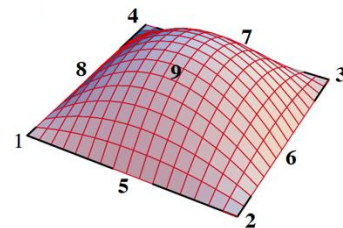
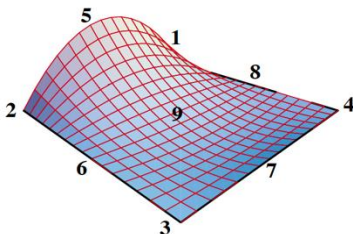
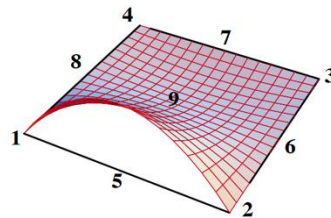
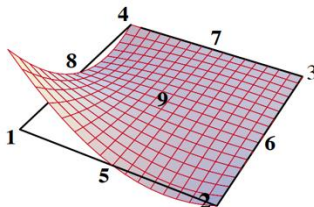


2-D 6-node
triangle

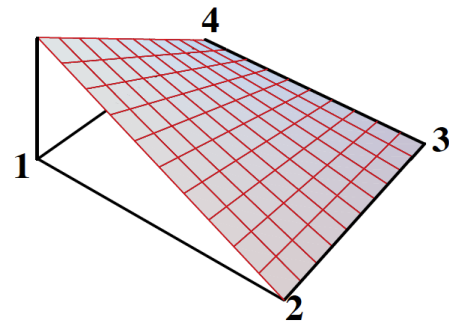


2-D 8-node
quadrilateral

Quadratic



Linear



Global numbering

Sparse matrix

$$\mathbf{K} = \begin{bmatrix} K_{11} & 0 & 0 & K_{14} & 0 & K_{16} \\ & K_{22} & 0 & 0 & K_{25} & 0 \\ & & K_{33} & 0 & 0 & K_{36} \\ & & & K_{44} & K_{45} & K_{46} \\ & \text{sym} & & & K_{55} & K_{56} \\ & & & & & K_{66} \end{bmatrix}$$

Band matrix

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 & 0 & 0 \\ & K_{22} & K_{23} & 0 & 0 & 0 \\ & & K_{33} & K_{34} & 0 & 0 \\ & & & K_{44} & K_{45} & K_{46} \\ & \text{sym} & & & K_{55} & K_{56} \\ & & & & & K_{66} \end{bmatrix}$$

Solution

Linearization

$$Ax - b = 0$$

Numerical techniques

~~$$x = A^{-1}b$$~~

- **Direct methods**
 - Gaussian elimination
 - LU decomposition
 - Cholesky
- **Iterative methods**
 - Jacobi
 - Gauss–Seidel
 - Conjugate Gradient (CG)

Theoretical basis

Differential equation
(Strong form)

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$$

$$\mathbf{u} = \hat{\mathbf{u}} \quad \text{on } \Gamma_u$$

$$\boldsymbol{\sigma} \mathbf{n} = \hat{\mathbf{t}} \quad \text{on } \Gamma_t$$

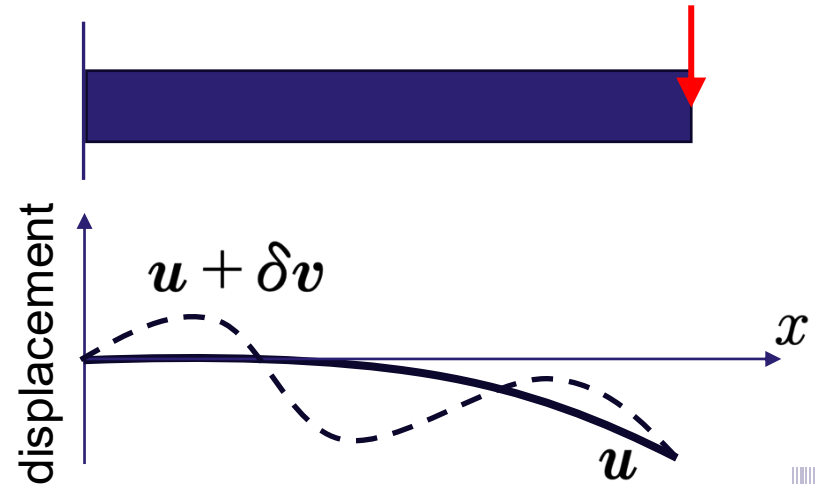
$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$$

$$\boldsymbol{\sigma} = \mathbb{C} \boldsymbol{\varepsilon}$$

second derivatives of displacement !!!

Variational formulation
(Weak form)

$$\int_{\Omega} \delta \mathbf{v} \cdot (\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{b}) d\Omega = 0$$



$$\mathbf{u} = \arg \min_{\mathbf{u} + \delta \mathbf{v}} \Pi(\mathbf{u} + \delta \mathbf{v})$$

Theoretical basis

Variational formulation
(Weak form)

$$\int_{\Omega} \delta \mathbf{v} \cdot (\nabla \cdot \boldsymbol{\sigma} + \mathbf{b}) d\Omega = 0$$

$$\delta \mathbf{v} \cdot (\nabla \cdot \boldsymbol{\sigma}) = \nabla \cdot (\boldsymbol{\sigma} \delta \mathbf{v}) - \boldsymbol{\sigma} : \nabla \delta \mathbf{v} \quad \text{product rule}$$

$$\int_{\Omega} [\delta \mathbf{v} \cdot \mathbf{b} + (\nabla \cdot (\boldsymbol{\sigma} \delta \mathbf{v}) - \boldsymbol{\sigma} : \nabla \delta \mathbf{v})] d\Omega = 0$$

$$\int_{\Omega} \nabla \cdot (\boldsymbol{\sigma} \delta \mathbf{v}) d\Omega = \int_{\partial\Omega} (\boldsymbol{\sigma} \delta \mathbf{v}) \cdot \mathbf{n} d\Gamma \quad \text{Gauss–Ostrogradsky (divergence) theorem}$$



Theoretical basis

Variational formulation
(Weak form)

$$\int_{\Omega} \nabla \cdot (\boldsymbol{\sigma} \delta \mathbf{v}) d\Omega = \int_{\partial\Omega} (\boldsymbol{\sigma} \delta \mathbf{v}) \cdot \mathbf{n} d\Gamma$$

$$\int_{\Omega} [\delta \mathbf{v} \cdot \mathbf{b} + (\nabla \cdot (\boldsymbol{\sigma} \delta \mathbf{v}) - \boldsymbol{\sigma} : \nabla \delta \mathbf{v})] d\Omega = 0$$

$$\int_{\Omega} \delta \mathbf{v} \mathbf{b} d\Omega - \int_{\Omega} \boldsymbol{\sigma} : \nabla \delta \mathbf{v} d\Omega + \int_{\partial\Omega} \delta \mathbf{v} \cdot (\boldsymbol{\sigma} \mathbf{n}) d\Gamma = 0$$

$$\boldsymbol{\sigma} : \nabla \delta \mathbf{v} = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\delta \mathbf{v}) + \cancel{\boldsymbol{\sigma} : \boldsymbol{\omega}(\delta \mathbf{v})} = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\delta \mathbf{v})$$

$$\int_{\Omega} \delta \mathbf{v} \mathbf{b} d\Omega - \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\delta \mathbf{v}) d\Omega + \int_{\partial\Omega} \delta \mathbf{v} \cdot (\boldsymbol{\sigma} \mathbf{n}) d\Gamma = 0$$

Theoretical basis

Variational formulation
(Weak form)

$$\int_{\Omega} \delta \mathbf{v} \mathbf{b} d\Omega - \int_{\Omega} \boldsymbol{\sigma} : \delta \boldsymbol{\epsilon} d\Omega + \int_{\partial\Omega} \delta \mathbf{v} \cdot (\boldsymbol{\sigma} \mathbf{n}) d\Gamma = 0$$

$$\delta W_{\text{int}} - \delta W_{\text{ext}} = 0 \quad \text{virtual work}$$

$$\delta W_{\text{int}} = \int_{\Omega} \boldsymbol{\sigma} : \delta \boldsymbol{\epsilon} d\Omega$$

$$\delta W_{\text{ext}} = \int_{\Omega} \delta \mathbf{v} \cdot \mathbf{b} d\Omega + \int_{\partial\Omega} \delta \mathbf{v} \cdot (\boldsymbol{\sigma} \mathbf{n}) d\Gamma$$



Theoretical basis

Energy of the system (linear elasticity)

$$\psi = \int_{\boldsymbol{\varepsilon}} \boldsymbol{\sigma} : d\boldsymbol{\varepsilon} = \frac{1}{2} \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} \quad U = \int_{\Omega} \psi(\boldsymbol{\varepsilon}(\boldsymbol{u})) d\Omega$$

$$W_{ext} = \int_{\Omega} \boldsymbol{u} \cdot \boldsymbol{b} d\Omega + \int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{t} d\Gamma = 0 \quad \Pi(\boldsymbol{u}) = U(\boldsymbol{u}) + W_{ext}(\boldsymbol{u})$$

$$\Pi(\boldsymbol{u}) = \frac{1}{2} \int_{\Omega} \boldsymbol{\varepsilon}(\boldsymbol{u}) : \mathbb{C} : \boldsymbol{\varepsilon}(\boldsymbol{u}) d\Omega - W_{ext}(\boldsymbol{u})$$

$$\boldsymbol{u} = \arg\min_w \Pi(\boldsymbol{w})$$

Theoretical basis

Energy of the system (linear elasticity)

$$\boldsymbol{u} = \arg \min_w \Pi(\boldsymbol{w}) \quad \boldsymbol{w} = \boldsymbol{u} + \delta \boldsymbol{v}$$

$$\Pi(\boldsymbol{u} + \delta \boldsymbol{v}) = \frac{1}{2} \int_{\Omega} \boldsymbol{\varepsilon}(\boldsymbol{u} + \delta \boldsymbol{v}) : \mathbb{C} : \boldsymbol{\varepsilon}(\boldsymbol{u} + \delta \boldsymbol{v}) d\Omega - W_{\text{ext}}(\boldsymbol{u} + \delta \boldsymbol{v})$$

$$\boldsymbol{\varepsilon}(\boldsymbol{u} + \delta \boldsymbol{v}) = \boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}(\delta \boldsymbol{v}) \quad \text{small deformation}$$

$$U(\boldsymbol{u} + \delta \boldsymbol{v}) = \frac{1}{2} \int_{\Omega} \boldsymbol{\varepsilon}(\boldsymbol{u}) : \mathbb{C} : \boldsymbol{\varepsilon}(\boldsymbol{u}) d\Omega + \int_{\Omega} \boldsymbol{\varepsilon}(\delta \boldsymbol{v}) : \mathbb{C} : \boldsymbol{\varepsilon}(\boldsymbol{u}) d\Omega + \frac{1}{2} \int_{\Omega} \boldsymbol{\varepsilon}(\delta \boldsymbol{v}) : \mathbb{C} : \boldsymbol{\varepsilon}(\delta \boldsymbol{v}) d\Omega$$

$$W_{\text{ext}}(\boldsymbol{u} + \delta \boldsymbol{v}) = W_{\text{ext}}(\boldsymbol{u}) + W_{\text{ext}}(\delta \boldsymbol{v})$$

Theoretical basis

Energy of the system (linear elasticity)

$$\begin{aligned}\Pi(u + v) &= \underbrace{\frac{1}{2} \int_{\Omega} \varepsilon(u) : \mathbb{C} : \varepsilon(u) d\Omega - W_{\text{ext}}(u)}_{\Pi(u)} \\ &\quad + \underbrace{\int_{\Omega} \varepsilon(\delta v) : \mathbb{C} : \varepsilon(u) d\Omega - W_{\text{ext}}(\delta v)}_{\delta \Pi(u; v) = 0} \\ &\quad + \frac{1}{2} \int_{\Omega} \varepsilon(\delta v) : \mathbb{C} : \varepsilon(\delta v) d\Omega\end{aligned}$$

Theoretical basis

Energy of the system (linear elasticity)

$$\Pi(u + \delta v) - \Pi(u) = \frac{1}{2} \int_{\Omega} \varepsilon(\delta v) : \mathbb{C} : \varepsilon(\delta v) d\Omega$$

$$\Pi(u + \delta v) - \Pi(u) > 0$$

$$\Pi(u + \delta v) > \Pi(u)$$

$$u = \operatorname{argmin}_w \Pi(w)$$

Theoretical basis

Finite element approximation

$$u = \arg \min_w \Pi(w)$$

$$u(\mathbf{X}) = \mathbf{N}(\mathbf{X}) \mathbf{U}$$

shape functions

DOFs

$$\Pi(\mathbf{U}) = \frac{1}{2} \mathbf{U}^T \mathbf{K} \mathbf{U} - \mathbf{U}^T \mathbf{F}$$

$$\frac{\partial \Pi}{\partial \mathbf{U}} = 0 \quad \Rightarrow \quad \mathbf{K} \mathbf{U} - \mathbf{F} = 0$$

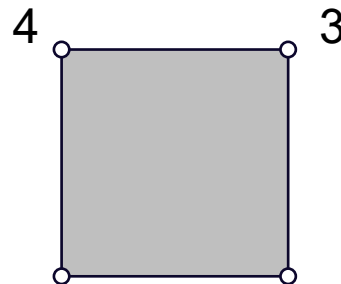
Tangent

Residual

Implementation

Elementary matrices

$$u = \arg \min_w \Pi(w)$$



$$U = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

$$u(X) = N(X)U$$

$$F_{\text{int}} = \frac{\partial U}{\partial U} = \frac{\partial}{\partial U} \frac{1}{2} \int_{\Omega} \sigma : \epsilon(u(U)) d\Omega$$

$$= \frac{\partial}{\partial U} \frac{1}{2} \int_{\Omega} \epsilon^T \mathbb{C} \epsilon d\Omega \quad \leftarrow \quad \epsilon = \frac{1}{2} (\nabla u + \nabla u^T) = \nabla^s N U = B U$$

$$\epsilon_x = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3 + \frac{\partial N_4}{\partial x} u_4$$

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} \\ \frac{\partial N_3}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_2}{\partial x} \end{bmatrix}$$

Implementation

Elementary matrices

$$\mathbf{F} = \mathbf{F}_{ext} - \mathbf{F}_{int}$$

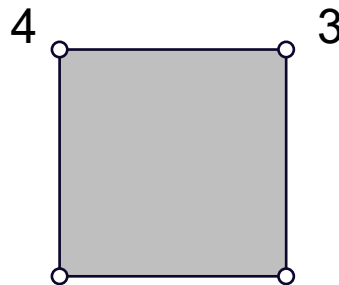
$$\mathbf{F}_{int} = \frac{\partial U}{\partial \mathbf{U}} = \frac{\partial}{\partial \mathbf{U}} \frac{1}{2} \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\epsilon}(\mathbf{u}(\mathbf{U})) d\Omega$$

$$= \frac{\partial}{\partial \mathbf{U}} \frac{1}{2} \int_{\Omega} \boldsymbol{\epsilon}^T \mathbb{C} \boldsymbol{\epsilon} d\Omega$$

$$= \frac{\partial}{\partial \mathbf{U}} \frac{1}{2} \int_{\Omega} (\mathbf{B}\mathbf{U})^T \mathbb{C} (\mathbf{B}\mathbf{U}) d\Omega$$

$$= \int_{\Omega} \mathbf{B}^T \mathbb{C} \cdot (\mathbf{B}\mathbf{U}) d\Omega = \int_{\Omega} \mathbf{B}^T \boldsymbol{\sigma} d\Omega$$

$$\mathbf{u} = \arg \min_w \Pi(\mathbf{w})$$



$$\mathbf{U} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

Implementation

Elementary matrices

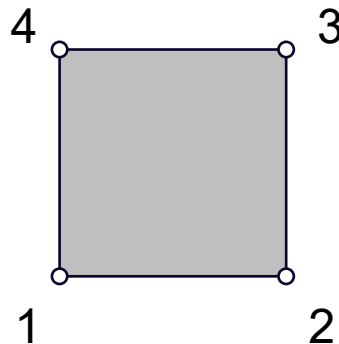
$$\mathbf{F}_{\text{int}} = \int_{\Omega} \mathbf{B}^T \boldsymbol{\sigma} d\Omega$$

$$\boldsymbol{\sigma} = \mathbb{C} \boldsymbol{\epsilon}$$

$$\mathbf{K} = \frac{\partial^2 U}{\partial \mathbf{U}^2}$$

$$\mathbf{K} = \frac{\partial^2}{\partial \mathbf{U}^2} \frac{1}{2} \int_{\Omega} (\mathbf{B}\mathbf{U})^T \mathbb{C} (\mathbf{B}\mathbf{U}) d\Omega = \int_{\Omega} \mathbf{B}^T \mathbb{C} \mathbf{B} d\Omega$$

$$\mathbf{u} = \arg \min_w \Pi(\mathbf{w})$$



Implementation

Numerical integration (Gauss-Legendre quadrature)

$$\int_{-1}^1 f(\xi) d\xi = \sum_{j=1}^n f(\xi_j) w_j + O\left(\frac{d^{2n} f}{f \xi^{2n}}\right) \quad \text{perfect integration of polynome: } 2n - 1$$

Table 2.1 Gaussian Quadrature Abscissae and Weights for $\int_{-1}^1 f(\xi) d\xi = \sum_{j=1}^n f(\xi_j) w_j$

Order	j	ξ_j	w_j	
$n = 1$	1	0	2	
$n = 2$	1	$+1/\sqrt{3}$	1	
	2	$-1/\sqrt{3}$	1	
$n = 3$	1	$+\sqrt{0.6}$	5/9	
	2	0	8/9	
	3	$-\sqrt{0.6}$	5/9	
$n = 4$	1	$+\sqrt{((3+a)/7)}$	$0.5 - 1/(3a)$	$a = \sqrt{4.8}$
	2	$+\sqrt{((3-a)/7)}$	$0.5 + 1/(3a)$	
	3	$-\sqrt{((3-a)/7)}$	$0.5 + 1/(3a)$	
	4	$-\sqrt{((3+a)/7)}$	$0.5 - 1/(3a)$	
$n = 5$	1	$+\sqrt{b}$	$((5c-3)d/b)$	$a = \sqrt{1120}$ $b = (70+a)/126$ $c = (70-a)/126$ $d = 1/(15(c-b))$
	2	$+\sqrt{c}$	$((3-5b)d/c)$	
	3	0	$2 - 2(w_1 + w_2)$	
	4	$-\sqrt{c}$	$((3-5b)d/c)$	
	5	$-\sqrt{b}$	$((5c-3)d/b)$	

Example:

$$f(\xi) = \xi^3 + \frac{\xi^2}{2} + 3$$

$$\int_{-1}^1 \left(\xi^3 + \frac{\xi^2}{2} + 3 \right) d\xi = \left[\frac{\xi^4}{4} + \frac{\xi^3}{6} + 3\xi \right]_{-1}^1 = 6.3333$$

Implementation

Numerical integration (Gauss-Legendre quadrature)

$$\int_{-1}^1 f(\xi) d\xi = \sum_{j=1}^n f(\xi_j) w_j + O\left(\frac{d^{2n}f}{f\xi^{2n}}\right) \quad \text{perfect integration of polynome: } 2n - 1$$

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Example:

$$f(\xi) = \xi^3 + \frac{\xi^2}{2} + 3$$

$$\sum_{j=1}^2 f(\xi_j) w_j = f\left(\frac{1}{\sqrt{3}}\right) \cdot 1 + f\left(-\frac{1}{\sqrt{3}}\right) \cdot 1$$

$$\xi_j = \frac{1}{\sqrt{3}}; -\frac{1}{\sqrt{3}} \quad = 3.3591 + 2.9742$$

$$w_j = 1; 1$$

Zienkiewicz, Taylor & Fox:
The Finite Element Method for Solid and Structural Mechanics



Implementation

Numerical integration (Gauss-Legendre quadrature)

$$\int_{-1}^1 f(\xi) d\xi = \sum_{j=1}^n f(\xi_j) w_j + O\left(\frac{d^{2n}f}{f\xi^{2n}}\right) \quad \text{perfect integration of polynome: } 2n - 1$$

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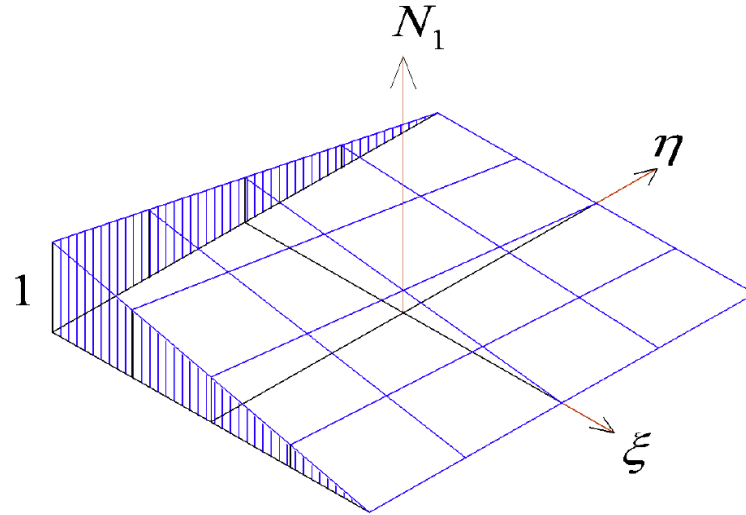
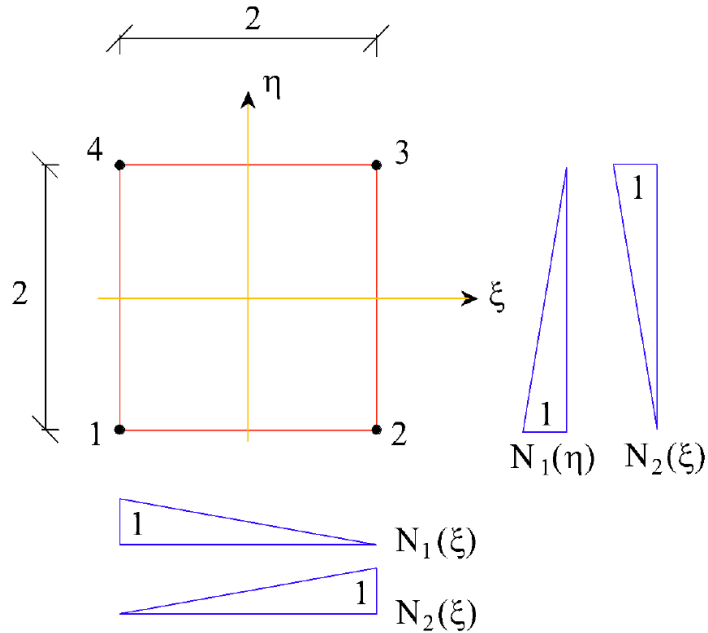
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$$w_j = 1; 1$$

Zienkiewicz, Taylor & Fox:
The Finite Element Method for Solid and Structural Mechanics

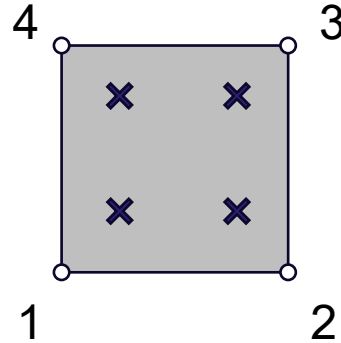
Implementation

Shape functions



$$N_1(\xi, \eta) = N_1(\xi)N_1(\eta)$$

$$u = \arg \min_w \Pi(w)$$



Implementation

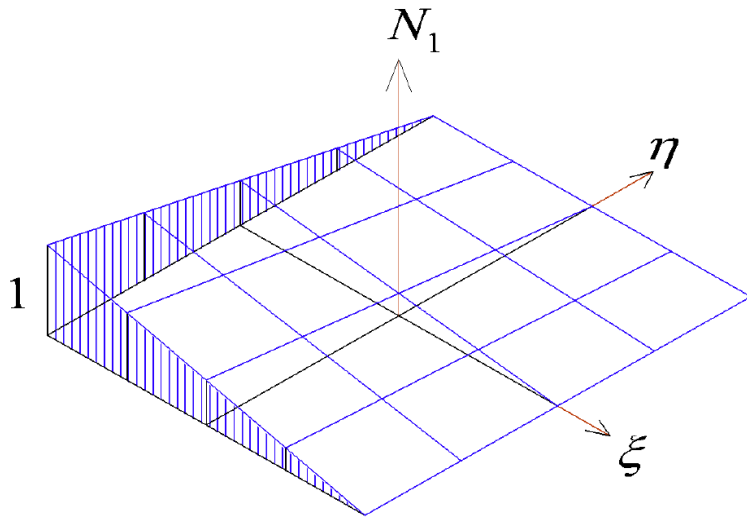
Shape functions

$$N_1 = \frac{1}{4}(1 - \xi)(1 - \eta)$$

$$N_2 = \frac{1}{4}(1 + \xi)(1 - \eta)$$

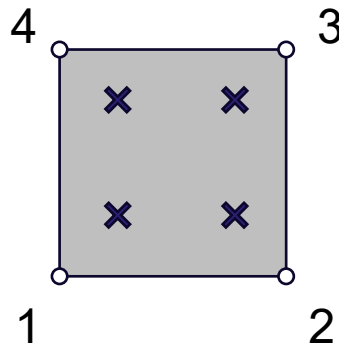
$$N_3 = \frac{1}{4}(1 + \xi)(1 + \eta)$$

$$N_4 = \frac{1}{4}(1 - \xi)(1 + \eta)$$



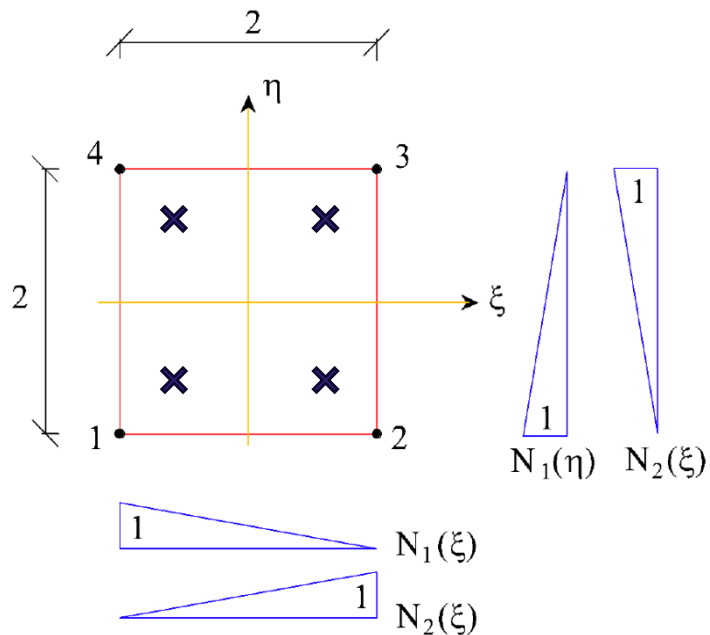
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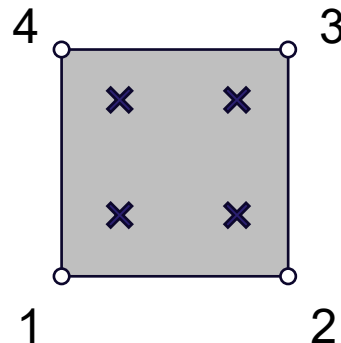
Implementation

Shape functions



$$[\xi, \eta] = \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$u = \arg \min_w \Pi(w)$$



Implementation

Spatial derivatives

$$\frac{\partial N_1}{\partial x} = \frac{\partial N_1}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial N_1}{\partial \eta} \frac{\partial \eta}{\partial x}$$

$$x = \sum_{i=1}^n x_i N_i \quad y = \sum_{i=1}^n y_i N_i$$

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^n x_i \frac{\partial N_i}{\partial \xi}$$

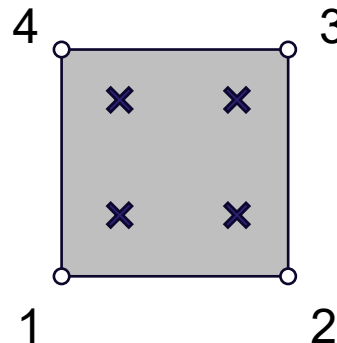
$$\frac{\partial y}{\partial \xi} = \sum_{i=1}^n y_i \frac{\partial N_i}{\partial \xi}$$

$$\frac{\partial x}{\partial \eta} = \sum_{i=1}^n x_i \frac{\partial N_i}{\partial \eta}$$

$$\frac{\partial y}{\partial \eta} = \sum_{i=1}^n y_i \frac{\partial N_i}{\partial \eta}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$u = \arg \min_w \Pi(w)$$



Jacobien

Implementation

Spatial derivatives

$$\frac{\partial N_1}{\partial x} = \frac{\partial N_1}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial N_1}{\partial \eta} \frac{\partial \eta}{\partial x}$$

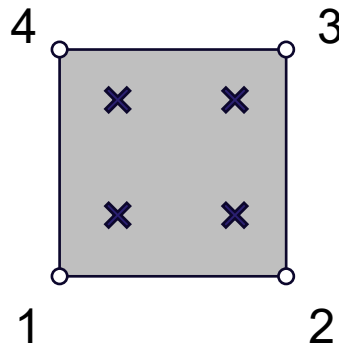
$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$J = \det(\mathbf{J})$ ← Integration point volume

$$\mathbf{J}^{-1} = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \frac{1}{\det(\mathbf{J})} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}$$

$$\nabla_{(x,y)} \mathbf{N} = \nabla_{(\xi,\eta)} \mathbf{N} \mathbf{J}^{-1}$$

$$u = \arg \min_w \Pi(w)$$



Implementation

Assembly

number of integration points

$$\mathbf{F}_{\text{int}}^e = \int_{\Omega^e} \mathbf{B}^T \boldsymbol{\sigma} d\Omega^e = \sum_{n=1}^p \mathbf{B}_n \boldsymbol{\sigma}_n J_n w_n$$

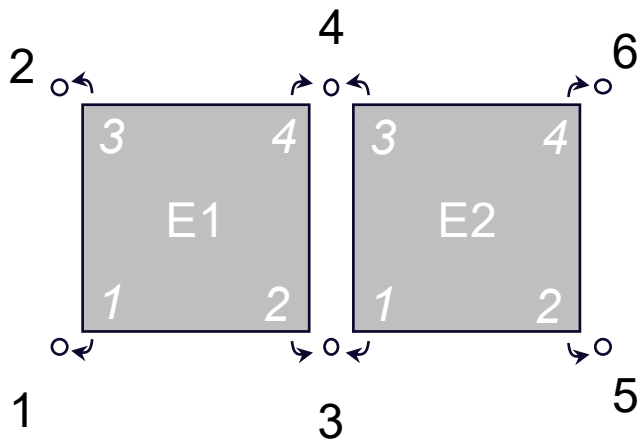
$$\mathbf{K}^e = \int_{\Omega} \mathbf{B}^T \mathbb{C} \mathbf{B} d\Omega = \sum_{n=1}^p \mathbf{B}_n^T \mathbb{C} \mathbf{B}_n J_n w_n$$

$$F_{(1,x)}^G = F_{(1,x)}^1$$

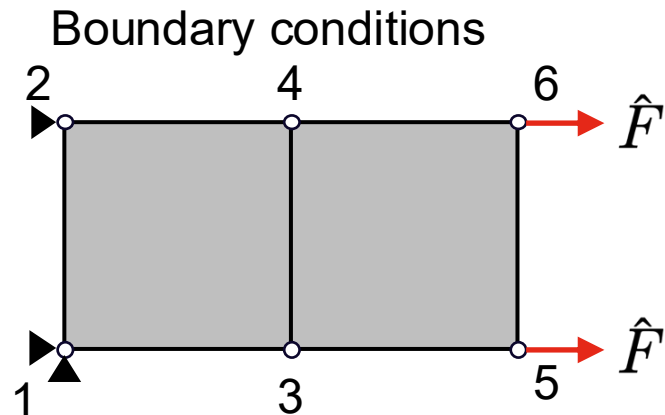
$$F_{(1,y)}^G = F_{(1,y)}^1$$

$$F_{(2,x)}^G = F_{(3,x)}^1$$

$$F_{(3,x)}^G = F_{(2,x)}^1 + F_{(1,x)}^2$$



Implementation



Neumann boundary conditions

$$(\mathbf{F}_{ext})_{(5,x)} = \mathbf{F}_{ext}[(5-1) \cdot 2 + 1] = \mathbf{F}_{ext}[9] = \hat{F}$$

$$(\mathbf{F}_{ext})_{(6,x)} = \mathbf{F}_{ext}[11] = \hat{F}$$

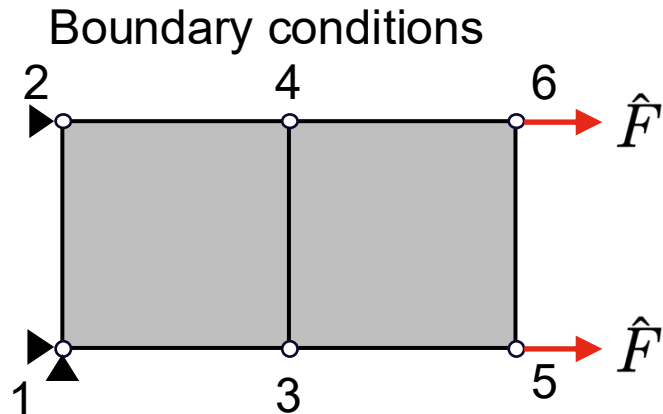
Dirichlet boundary conditions

Simple method: $\mathbf{K}^G[1,1] = 10^{10}$

$$\mathbf{K}^G[2,2] = 10^{10}$$

$$\mathbf{K}^G[3,3] = 10^{10}$$

Implementation



Neumann boundary conditions

$$(\mathbf{F}_{ext})_{(5,x)} = \mathbf{F}_{ext}[(5-1) \cdot 2 + 1] = \mathbf{F}_{ext}[9] = \hat{F}$$

$$(\mathbf{F}_{ext})_{(6,x)} = \mathbf{F}_{ext}[11] = \hat{F}$$

Dirichlet boundary conditions

Precise method:

$$U[J] = \hat{U}[J] \quad J \in \mathcal{D}$$

$$\mathbf{F}_{int}^G \leftarrow \mathbf{F}_{int}^G - \sum_{J \in \mathcal{D}} \mathbf{K}^G[:, J] \hat{U}[J]$$

$$\mathbf{K}^G[:, J] = 0 \quad \mathbf{F}_{int}^G[J] = -\hat{U}[J]$$

$$\mathbf{K}^G[J, :] = 0$$

$$\mathbf{K}^G[J, J] = 1$$

Implementation

Solution

$$\mathbf{F} = \mathbf{F}_{ext} - \mathbf{F}_{int}$$

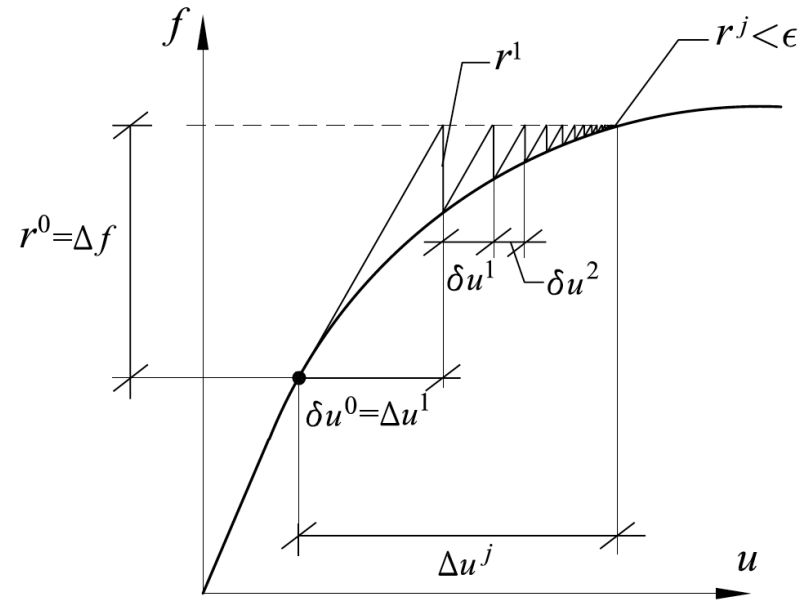
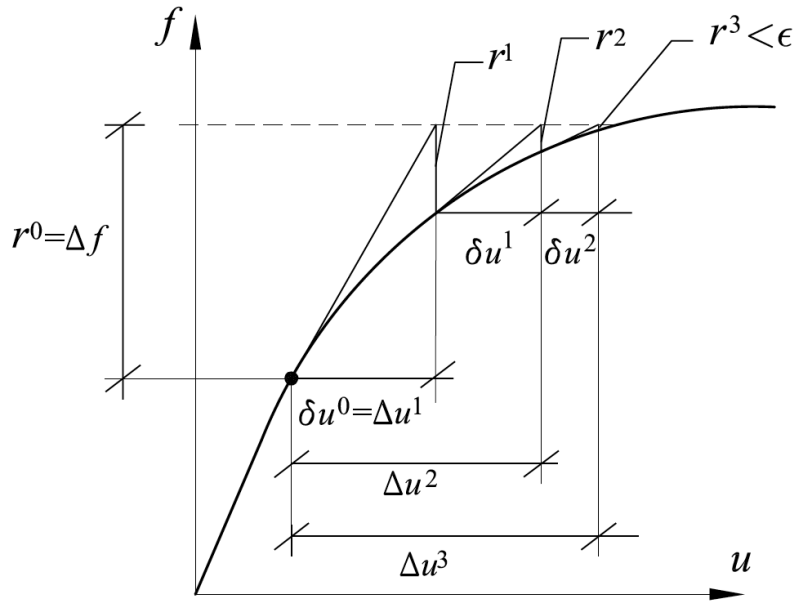
$$\mathbf{K}\mathbf{U} - \mathbf{F} = \mathbf{0}$$

$$(\mathbf{U} = \mathbf{K}^{-1}\mathbf{F})$$

$$\mathbf{U} = solve(\mathbf{K}, \mathbf{F})$$

Implementation

Convergence



Implementation

Secondary variables

$$\boldsymbol{\epsilon} = \boldsymbol{B}\boldsymbol{U}^e$$

$$\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\epsilon}$$

$$\psi = \frac{1}{2} \boldsymbol{\epsilon} : \boldsymbol{\sigma}$$

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