



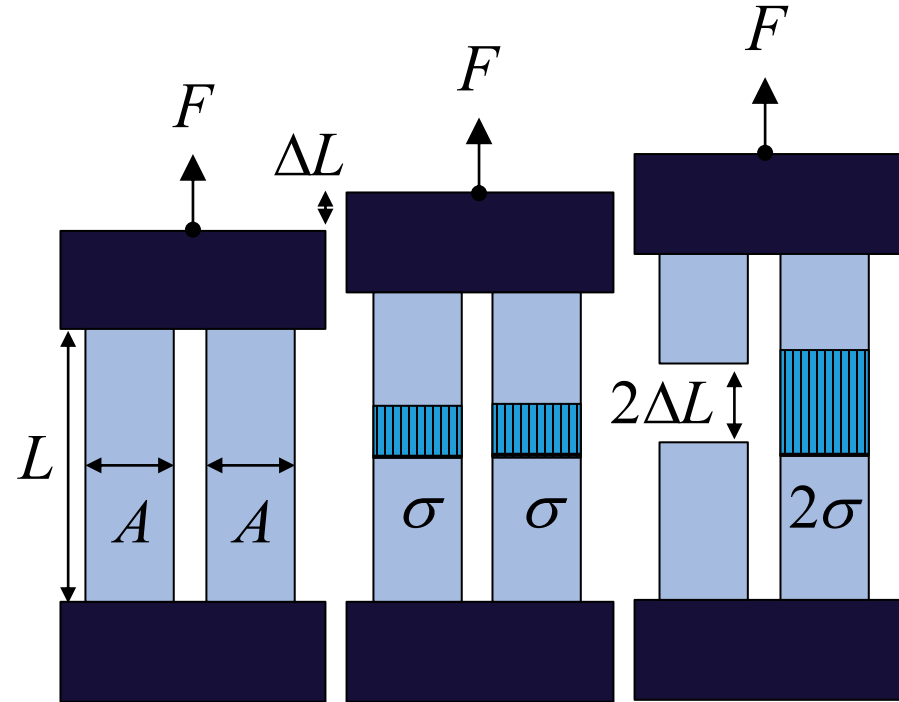
Fracture mechanics

Gergely Molnár



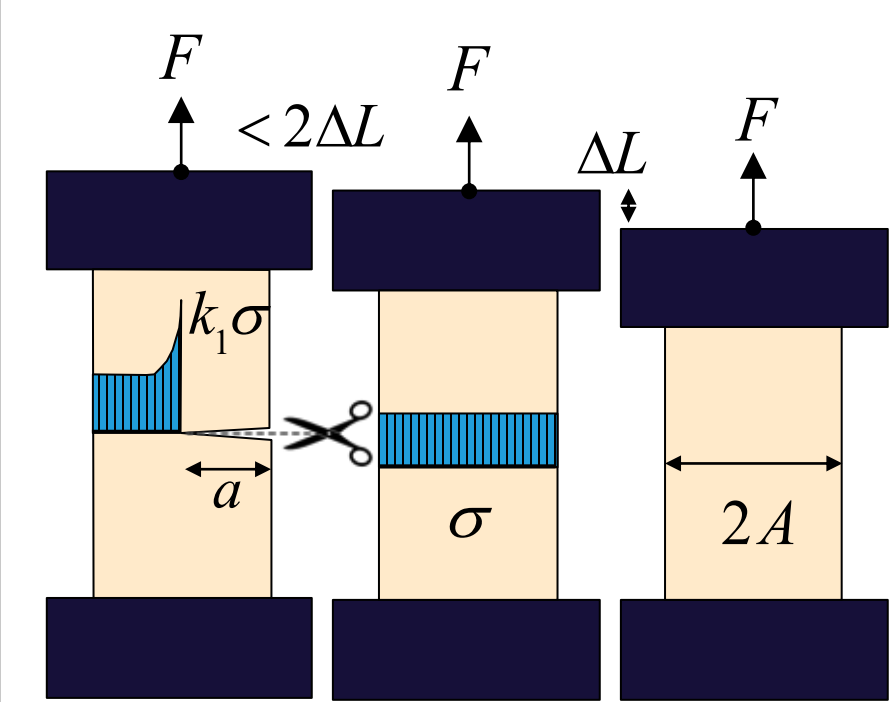
Motivation

Classical mechanics



$$\sigma = \frac{F}{2A}$$

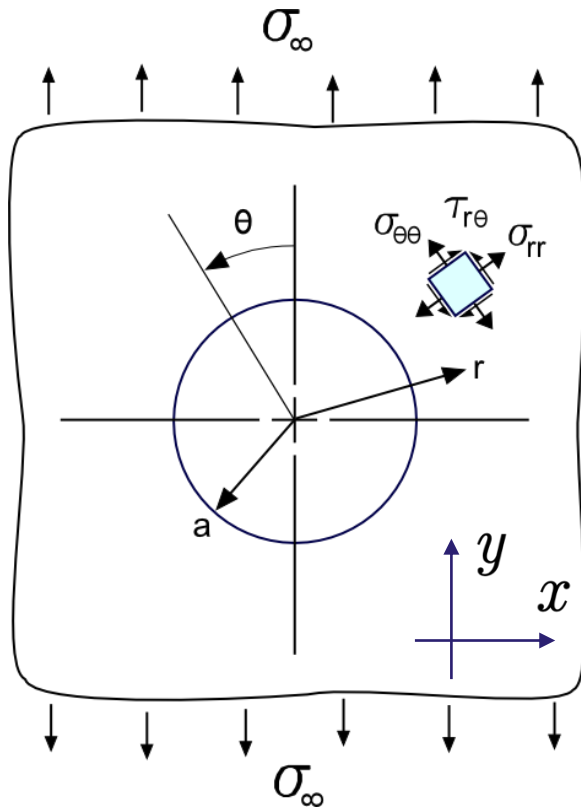
Fracture mechanics



$$\sigma = \frac{F\sqrt{\pi a}}{2A} \frac{1}{\sqrt{2\pi r}} g(\theta)$$

Elastic stress field

Circular inclusion (Airy stress function)



$$\nabla^4 \Phi = \nabla^2 (\nabla^2 \Phi) = 0$$

$$\nabla^2 \Phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}$$

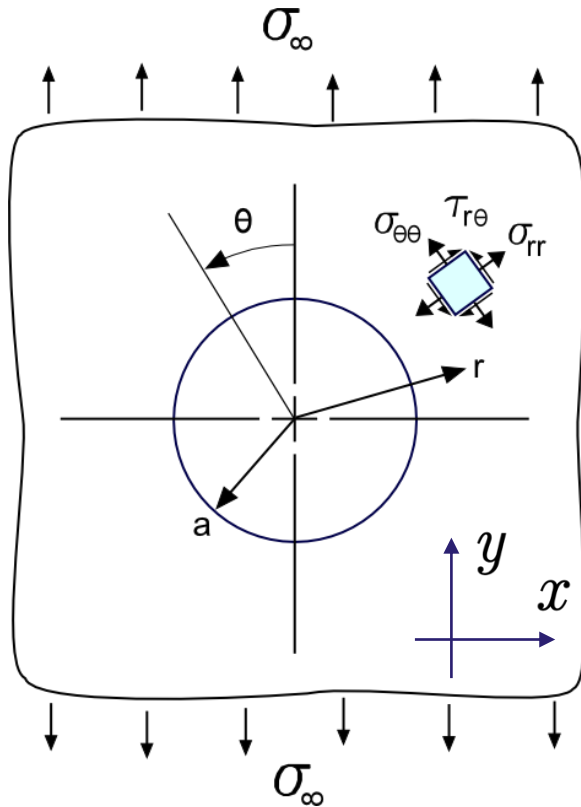
$$\sigma_{rr} = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right)$$

Elastic stress field

Circular inclusion (Airy stress function) – Boundary conditions



$$\sigma_{rr}(r = a, \theta) = 0$$

$$\tau_{r\theta}(r = a, \theta) = 0$$

$$\sigma_{r\theta} = Q^T \sigma Q$$

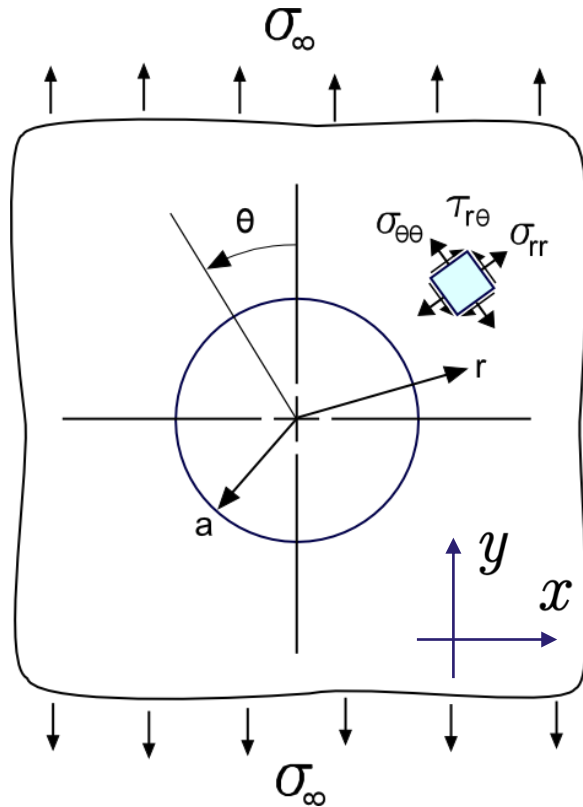
$$\sigma_{rr}(r = \infty, \theta) = \frac{\sigma_\infty}{2}(1 + \cos 2\theta)$$

$$\sigma_{\theta\theta}(r = \infty, \theta) = \frac{\sigma_\infty}{2}(1 - \cos 2\theta)$$

$$\tau_{r\theta}(r = \infty, \theta) = \frac{\sigma_\infty}{2} \sin 2\theta$$

Elastic stress field

Circular inclusion (Airy stress function)



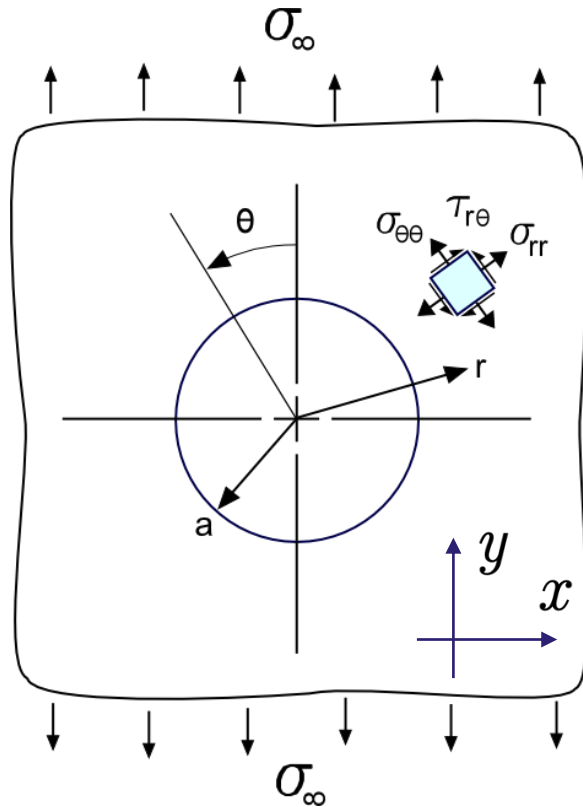
$$\sigma_{rr} = \frac{\sigma_\infty}{2} \left(1 - \frac{a^2}{r^2} \right) + \frac{\sigma_\infty}{2} \left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2} \right) \cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{\sigma_\infty}{2} \left(1 + \frac{a^2}{r^2} \right) - \frac{\sigma_\infty}{2} \left(1 + \frac{3a^4}{r^4} \right) \cos 2\theta$$

$$\tau_{r\theta} = -\frac{\sigma_\infty}{2} \left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2} \right) \sin 2\theta$$

Elastic stress field

Circular inclusion (Airy stress function)



$$\sigma_{rr}(r=a, \theta) = \frac{\sigma_{\infty}}{2}(1-1) + \frac{\sigma_{\infty}}{2}(1+3-4)\cos 2\theta = 0$$

$$\sigma_{\theta\theta}(r=a, \theta) = \frac{\sigma_{\infty}}{2}(1+1) - \frac{\sigma_{\infty}}{2}(1+3)\cos 2\theta$$

$$\tau_{r\theta}(r=a, \theta) = -\frac{\sigma_{\infty}}{2}(1-3+2)\sin 2\theta = 0$$

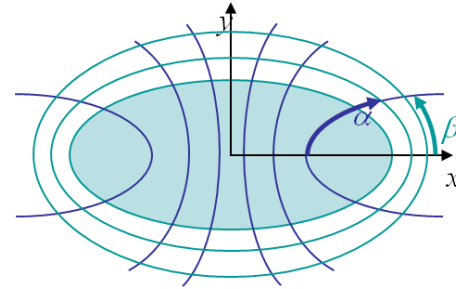
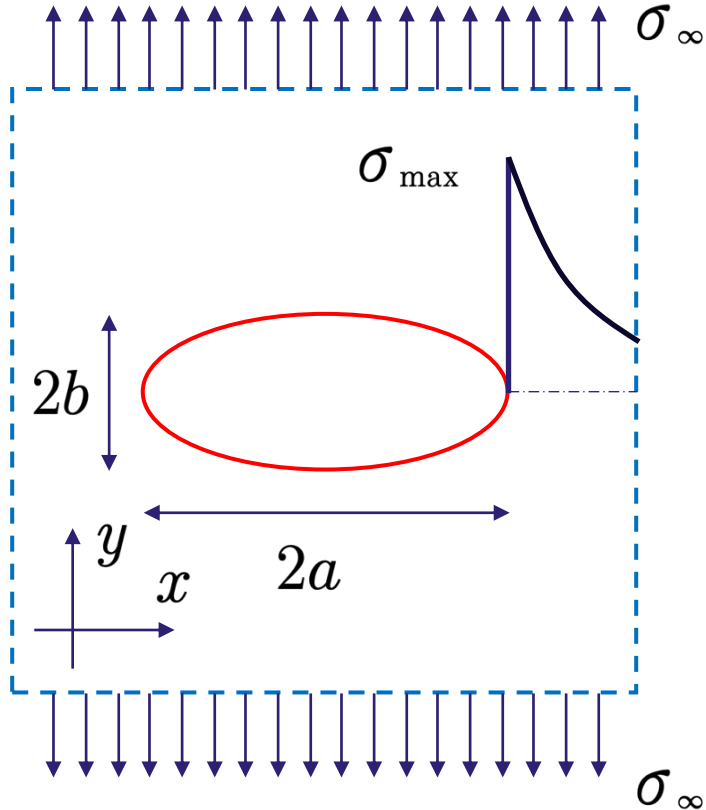
$$\sigma_{\theta\theta}(r=a, \theta) = \sigma_{\infty}(1-2\cos 2\theta)$$

$$\sigma_{\theta\theta}\left(r=a, \theta = \frac{\pi}{2}\right) = 3\sigma_{\infty}$$



Elastic stress field

Elliptic stress field (Inglis's solution)



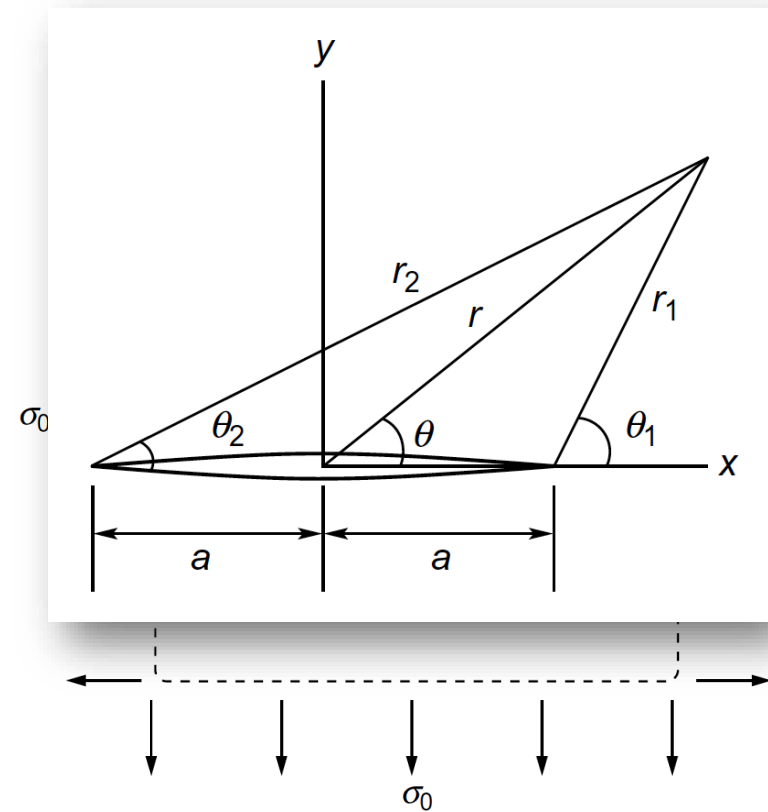
Inglis (1913)

$$\sigma_{\max} = \sigma_y(a, 0) = \sigma_\infty \left(1 + \frac{2a}{b} \right)$$

$$\frac{a}{b} \rightarrow \infty \implies \sigma_{\max} \rightarrow \infty$$

Elastic stress field

Crack (Westergaard's solution)



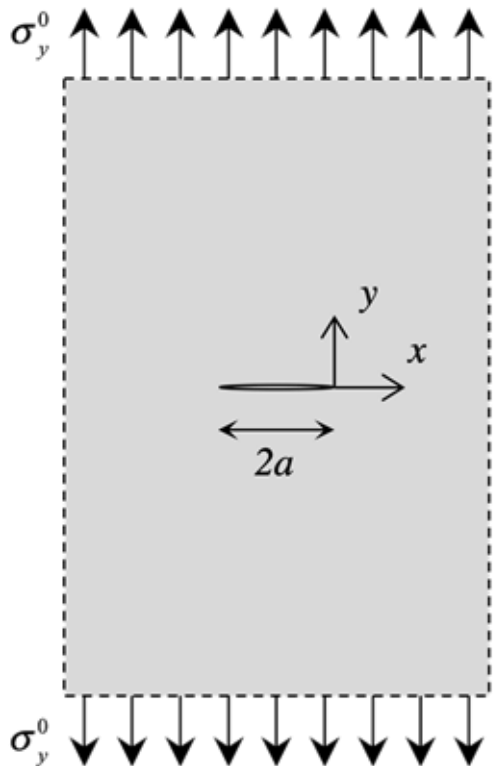
$$\sigma_x = \frac{\sigma_0 r}{\sqrt{r_1 r_2}} \left[\cos\left(\theta - \frac{1}{2}\theta_1 - \frac{1}{2}\theta_2\right) - \frac{a^2}{r_1 r_2} \sin\theta \sin\left\{\frac{3}{2}(\theta_1 + \theta_2)\right\} \right]$$

$$\sigma_y = \frac{\sigma_0 r}{\sqrt{r_1 r_2}} \left[\cos\left(\theta - \frac{1}{2}\theta_1 - \frac{1}{2}\theta_2\right) + \frac{a^2}{r_1 r_2} \sin\theta \sin\left\{\frac{3}{2}(\theta_1 + \theta_2)\right\} \right]$$

$$\tau_{xy} = \frac{\sigma_0 r}{\sqrt{r_1 r_2}} \left[\frac{a^2}{r_1 r_2} \sin\theta \cos\left\{\frac{3}{2}(\theta_1 + \theta_2)\right\} \right]$$

Energy release rate

Griffith's criterion



$$\pi(r=0, \theta) = \int_0^{\epsilon_i} \sigma_i d\epsilon_i = \infty$$

$$\Pi = \int_V \pi dV$$

$$\Delta\Pi(a) = \Pi(a) - \Pi(0) \neq \infty$$

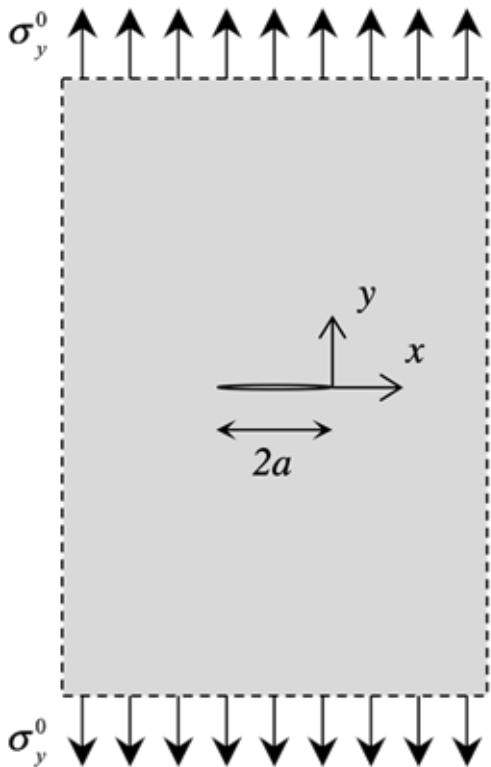
$$\Delta\Pi = \frac{\pi a^2}{2\mu} (1 - \nu) (\sigma_y^0)^2 h \quad \text{(plane strain)}$$



Griffith (1921, 1924)

Energy release rate

Griffith's criterion



$$\Delta\Pi = \frac{\pi a^2}{2\mu}(1-\nu)(\sigma_y^0)^2 h$$

$$S = 2g_c a h$$

$$\frac{\partial}{\partial a}(S - \Delta\Pi) = 0$$

$$\frac{1}{h} \frac{\partial \Delta\Pi}{\partial a} = G = \frac{\pi a}{2\mu}(1-\nu)(\sigma_y^0)^2 \geq g_c$$

$$\sigma_y^{0,cr} = \sqrt{\frac{2\mu g_c}{(1-\nu)a\pi}}$$

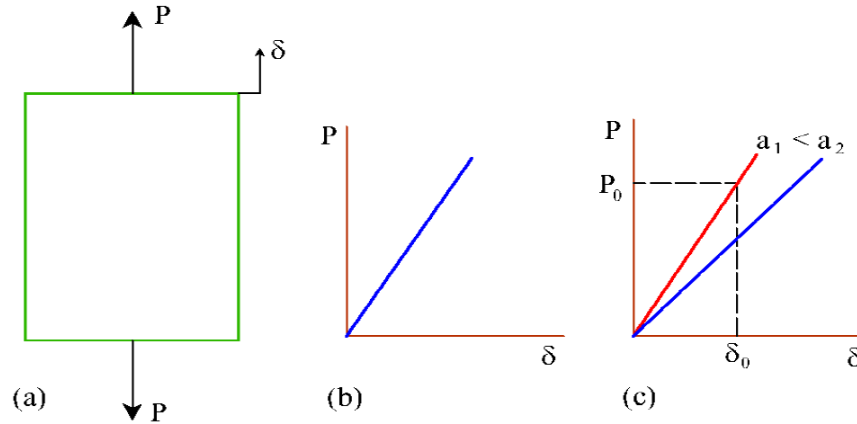


Griffith (1921, 1924)

$$g_c \left[\frac{J}{m^2} \right]$$

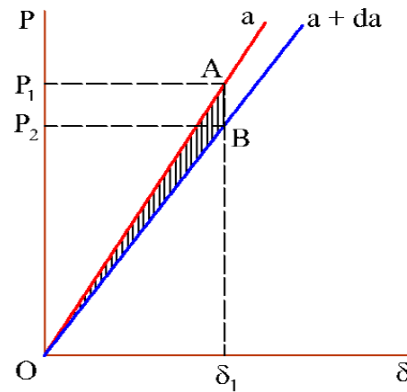
Energy release rate

Graphical interpretation

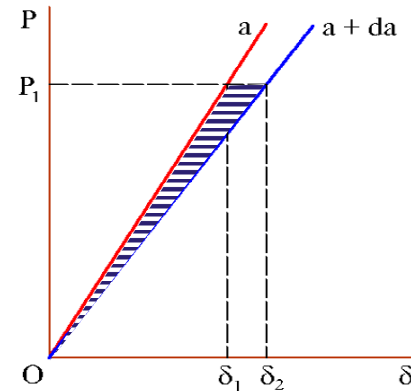


Griffith (1921, 1924)

Displacement controlled

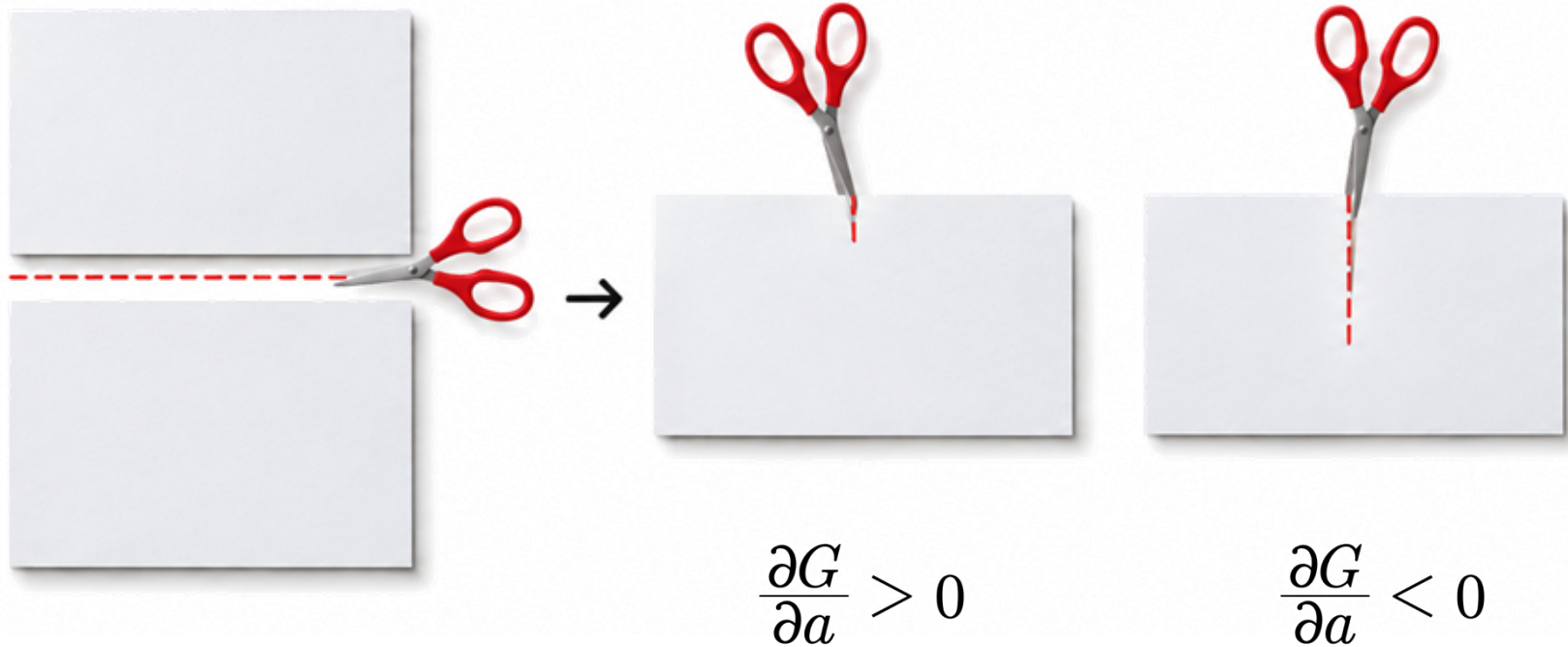


Force controlled



Energy release rate

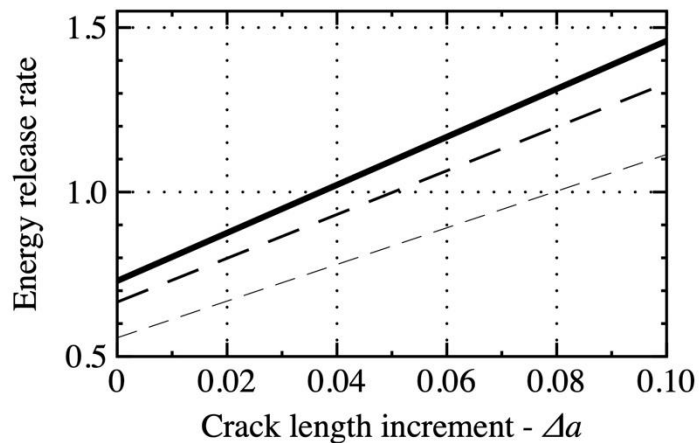
Stability



Energy release rate

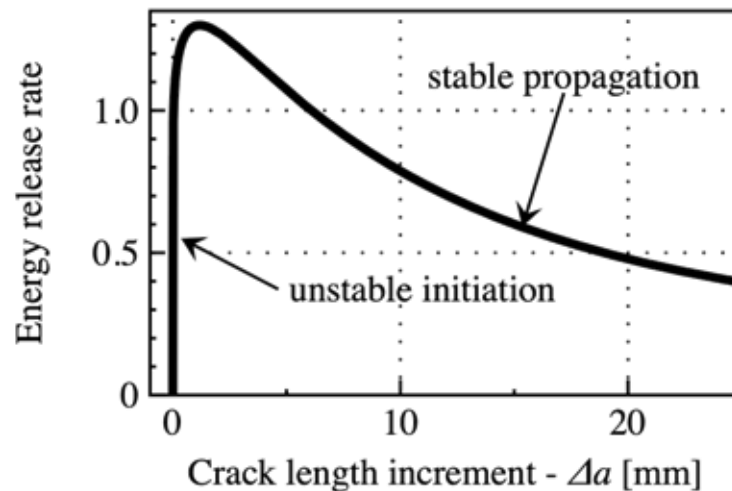
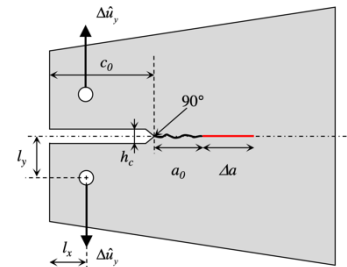
Stability

Unstable propagation



$$G = \frac{\pi a}{2\mu} (1 - \nu) (\sigma_y^0)^2 \quad \frac{\partial G}{\partial a} > 0$$

Stable propagation



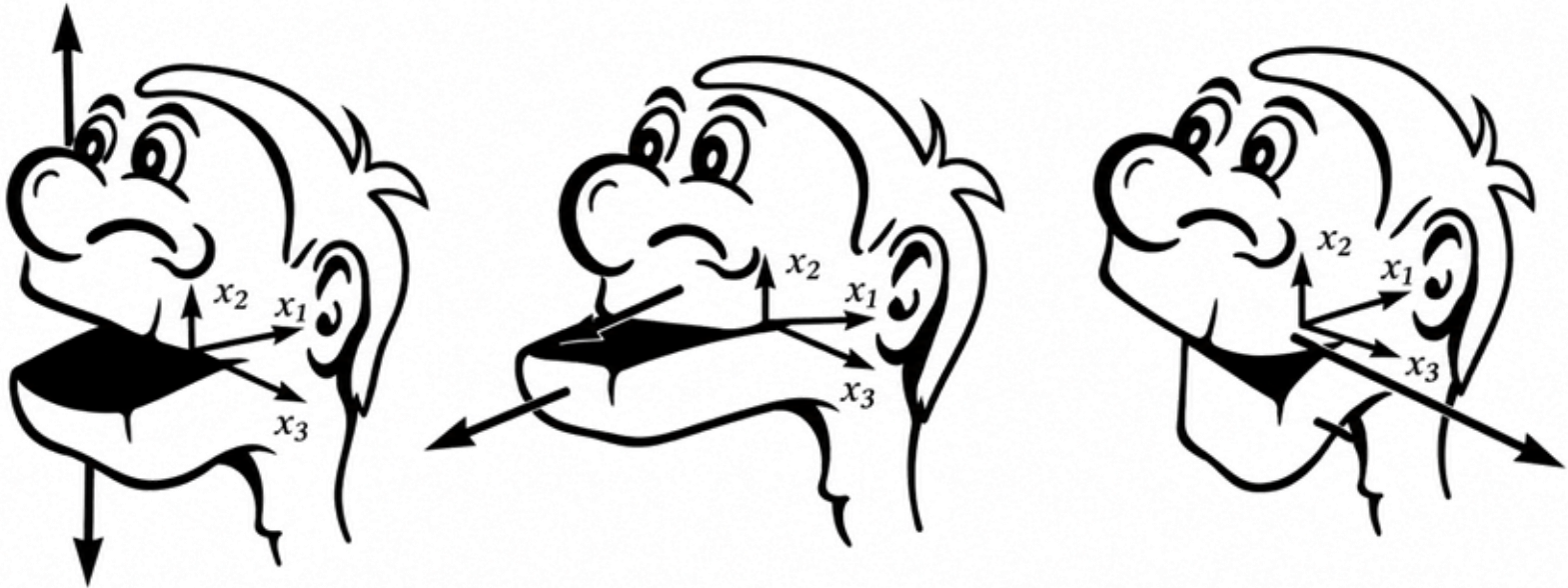
$$\frac{\partial G}{\partial a} < 0$$

Stress intensity factor

Modes of fracture

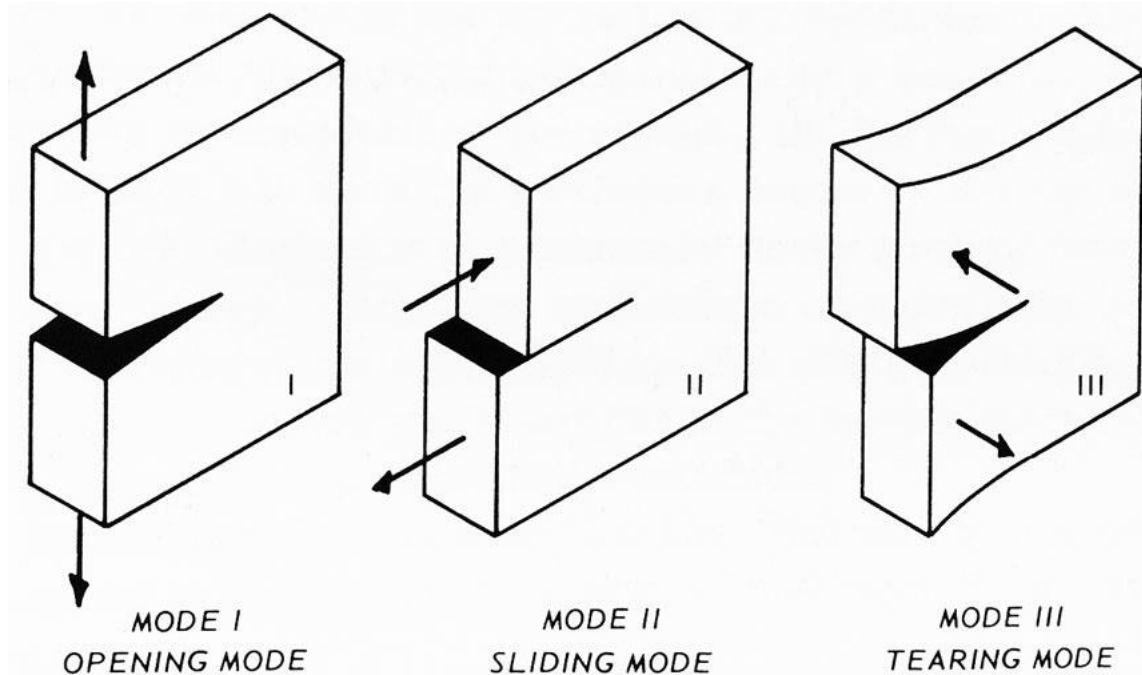


Irwin (1967, 1958)
McClintock & Irwin (1968)



Stress intensity factor

Modes of fracture



Irwin (1967, 1958)
McClintock & Irwin (1968)

Stress intensity factor

Near crack tip solution

(Semi-infinite Crack in an Infinite Medium)

$$\frac{r_1}{a} \ll 1 \quad \theta \approx \theta_2 \approx 0$$

$$r \approx a$$

$$r_2 \approx 2a$$

$$\sigma_x = \frac{\sigma_0 \sqrt{a}}{\sqrt{2r_1}} \cos\left(\frac{1}{2}\theta_1\right) \left[1 - \sin\left(\frac{1}{2}\theta_1\right) \sin\left(\frac{3}{2}\theta_1\right) \right]$$

$$\sigma_y = \frac{\sigma_0 \sqrt{a}}{\sqrt{2r_1}} \cos\left(\frac{1}{2}\theta_1\right) \left[1 + \sin\left(\frac{1}{2}\theta_1\right) \sin\left(\frac{3}{2}\theta_1\right) \right]$$

$$\tau_{xy} = \frac{\sigma_0 \sqrt{a}}{\sqrt{2r_1}} \sin\left(\frac{1}{2}\theta_1\right) \cos\left(\frac{1}{2}\theta_1\right) \cos\left(\frac{3}{2}\theta_1\right)$$



Stress intensity factor

Near crack tip solution

(Semi-infinite Crack in an Infinite Medium)

$$\frac{r_1}{a} \ll 1 \quad \theta \approx \theta_2 \approx 0$$

$$r \approx a$$

$$r_2 \approx 2a$$



Irwin (1967, 1958)
McClintock & Irwin (1968)

$$K_I = \sigma_0 \sqrt{a\pi}$$

$$K_c \text{ [Pa}\sqrt{\text{m}}\text{]}$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r_1}} \cos\left(\frac{1}{2}\theta_1\right) \left[1 + \sin\left(\frac{1}{2}\theta_1\right) \sin\left(\frac{3}{2}\theta_1\right) \right]$$

$$\sigma_{ij}(r, \theta) = \sum_{k=I,II,III} \frac{K_k}{\sqrt{2\pi r}} f_{k,ij}(\theta)$$

$$K \leq K_c$$



Stress intensity factor

Connection to Griffith

$$G = \frac{\pi a}{2\mu} (1 - \nu) \sigma_0^2$$

$$K_I = \sigma_0 \sqrt{a\pi}$$

$$a\sigma_0^2 = \frac{2\mu G}{\pi(1 - \nu)}$$

$$\frac{K_I^2}{\pi} = \sigma_0^2 a$$

$$\frac{2\mu G}{\pi(1 - \nu)} = \frac{K_I^2}{\pi}$$

$$G = \frac{K_I^2 (1 - \nu)}{2\mu} = \frac{K_I^2 (1 - \nu^2)}{E}$$

plane strain

Stress intensity factor

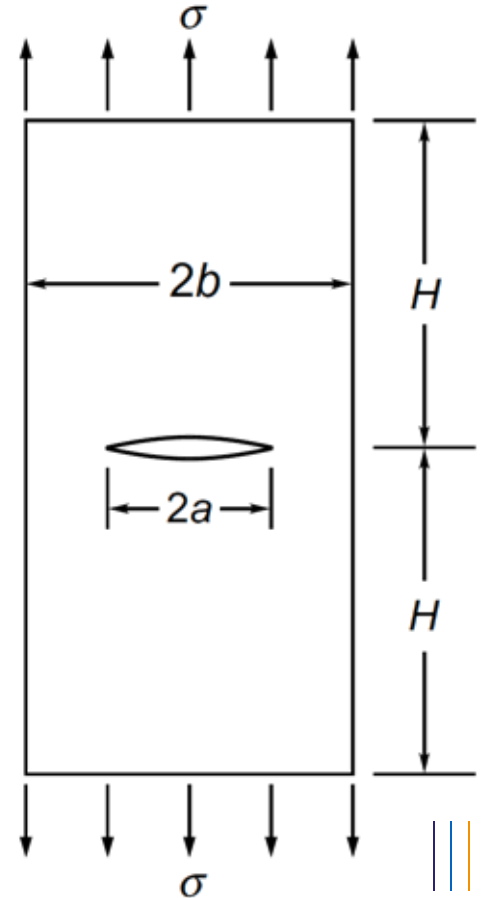
Finite size

$$K_I \neq \sigma \sqrt{a\pi}$$

$$K_I = \sigma \sqrt{a\pi} F\left(\frac{a}{b}, \frac{a}{H}\right)$$

$$F\left(\frac{a}{b}\right) = 1 + 0.128\left(\frac{a}{b}\right) - 0.288\left(\frac{a}{b}\right)^2 + 1.525\left(\frac{a}{b}\right)^3$$

$$\frac{a}{b} < 0.7$$



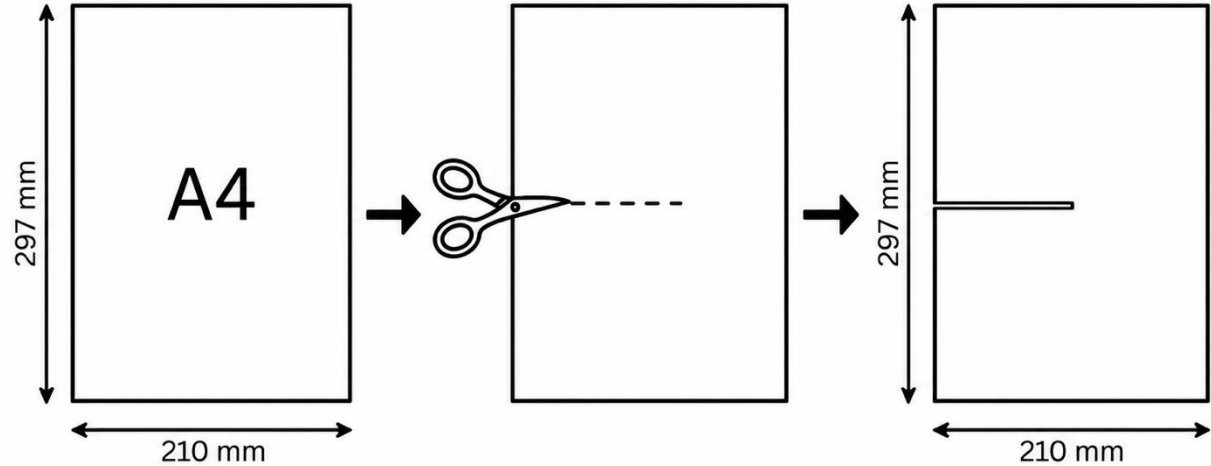
Stress intensity factor

Example

$$L = 210 \text{ mm}$$

$$t = 0.1 \text{ mm (75 g/m}^2\text{)}$$

$$a = 105 \text{ mm}$$



$$K_I = \frac{F}{L \cdot t} \sqrt{\pi a} \left[1.122 - 0.231 \left(\frac{a}{L} \right) + 10.55 \left(\frac{a}{L} \right)^2 - 21.71 \left(\frac{a}{L} \right)^3 + 30.382 \left(\frac{a}{L} \right)^4 \right]$$

$$K_{IC} = 3.2 \text{ MPa} \sqrt{\text{mm}}$$

Stress intensity factor

Williams' Series Expansion

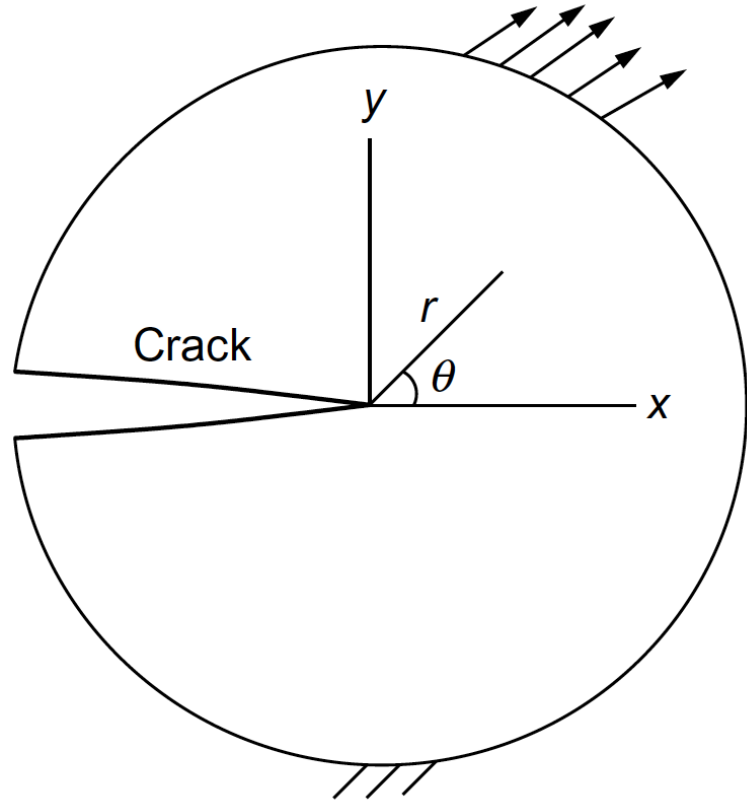
$$\sigma_{ij}(r, \theta) = \sum_{n=1}^{\infty} A_n r^{n/2-1} f_{ij}^{(n)}(\theta)$$

$$u_i(r, \theta) = \sum_{n=1}^{\infty} A_n r^{n/2} g_i^{(n)}(\theta)$$

$$A_1 = K_I$$

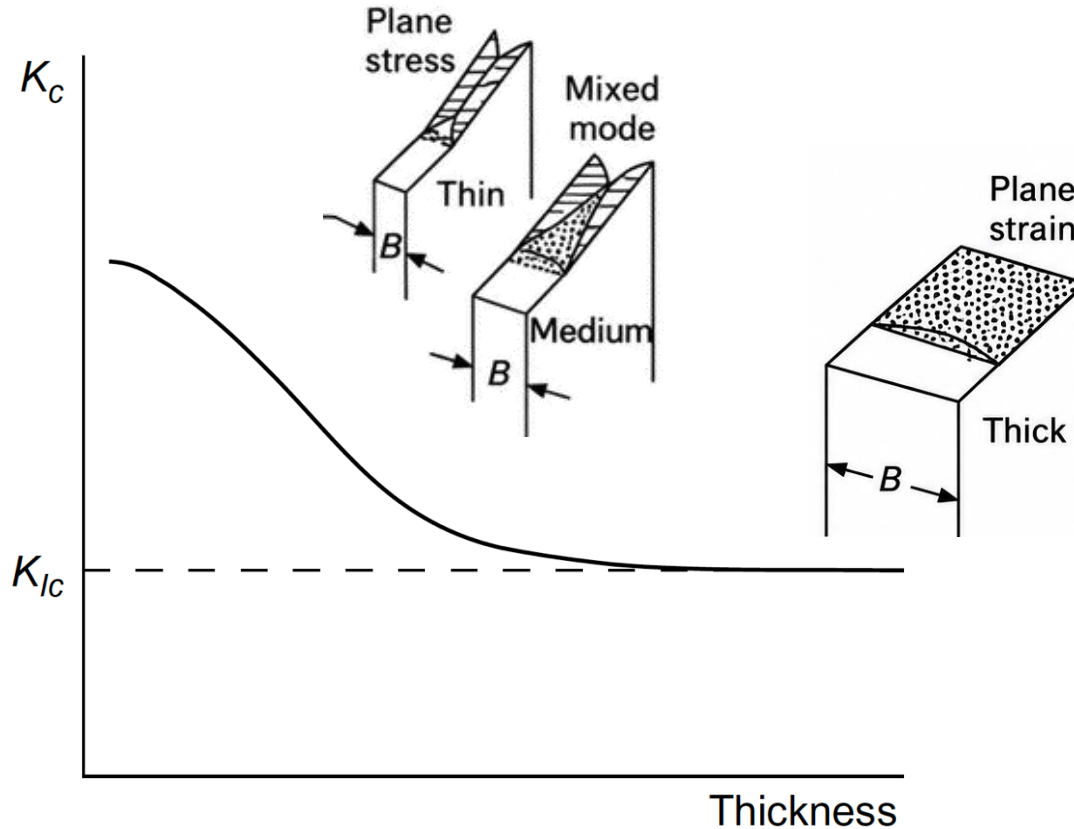
$$A_2 = T$$

$$A_3 = B$$



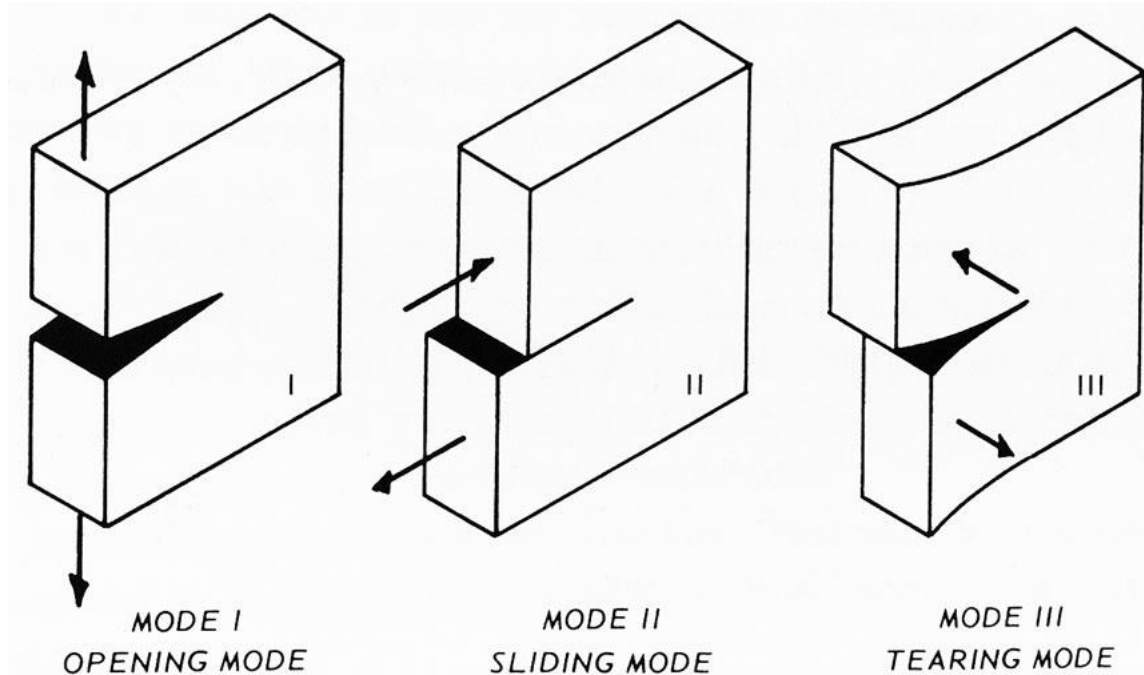
Stress intensity factor

Plane strain vs plane stress



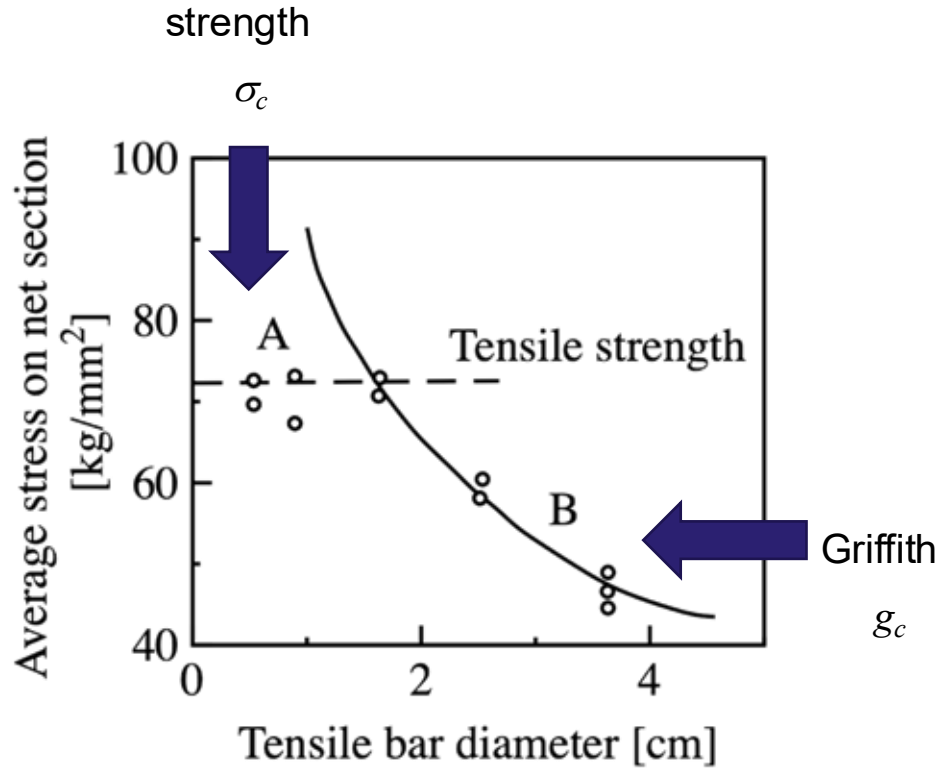
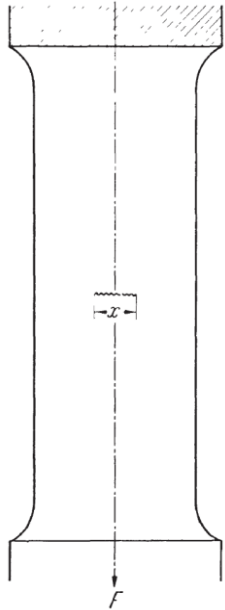
Stress intensity factor

Limitations



Stress intensity factor

Limitations

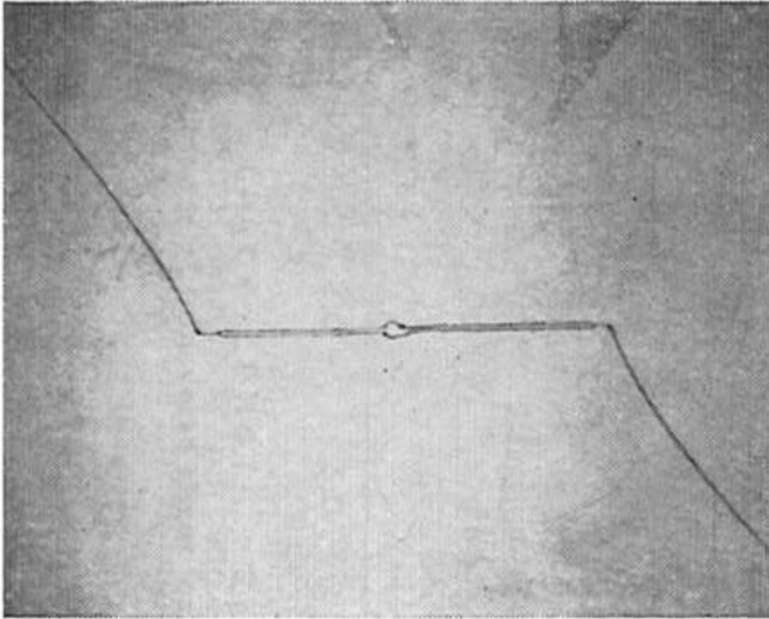


Irwin's length

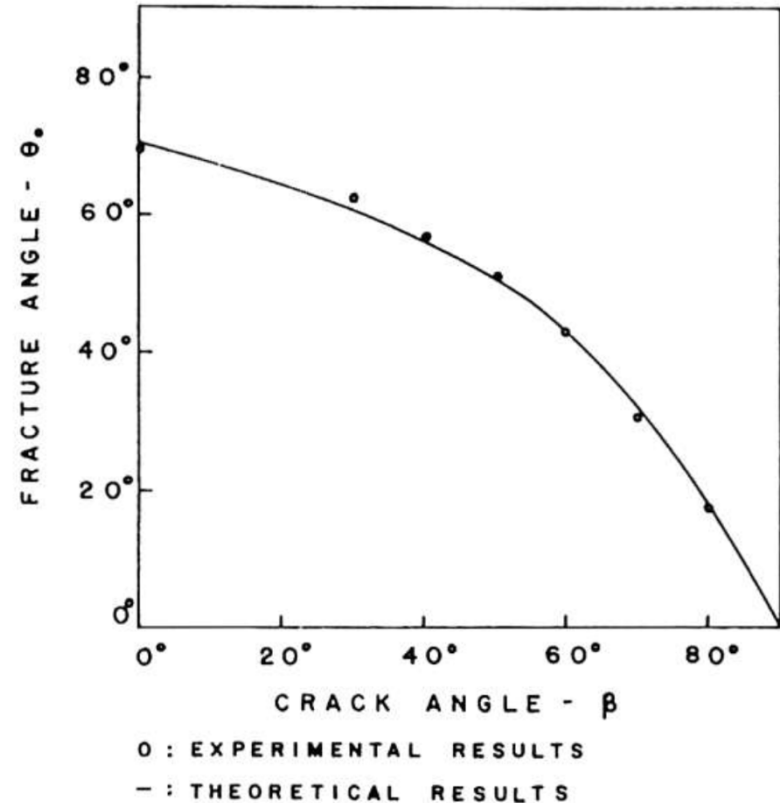
$$l_{mat} = \frac{Eg_c}{\sigma_c^2}$$

Stress intensity factor

Mode II

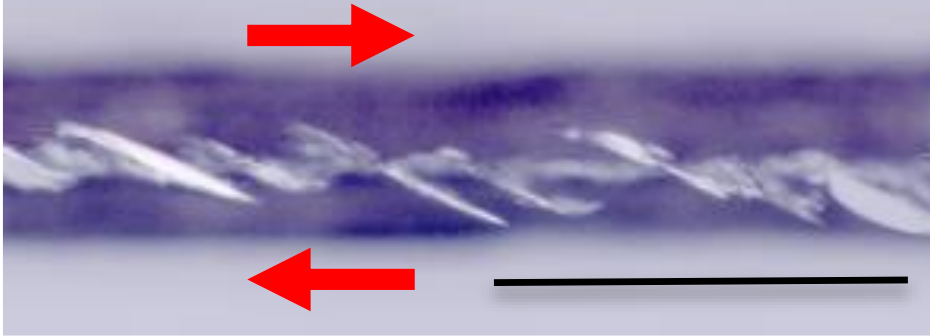


(Erdogan & Sih, 1963)

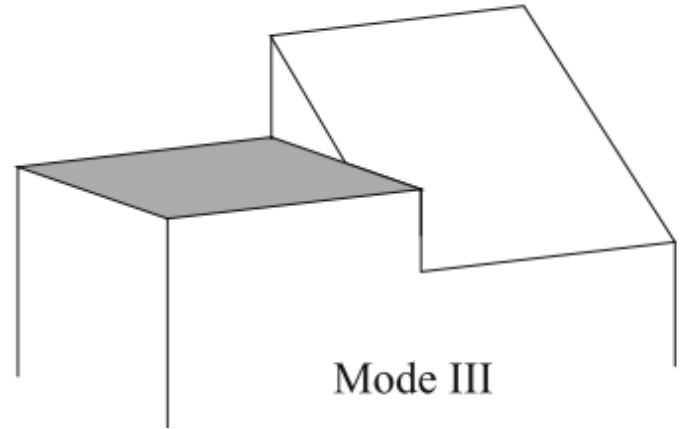


Theoretical basics

Mode III



(Sommer, 1969)
(Chen et al., 2015)

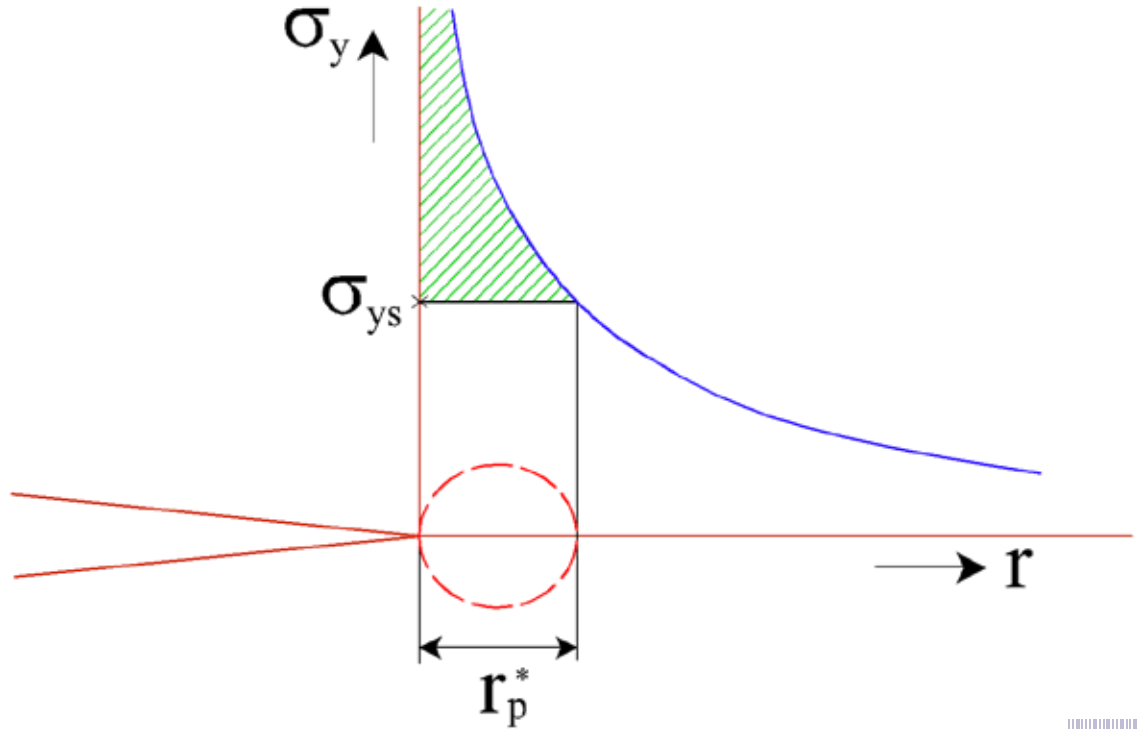


Ductile fracture

Irwin's model

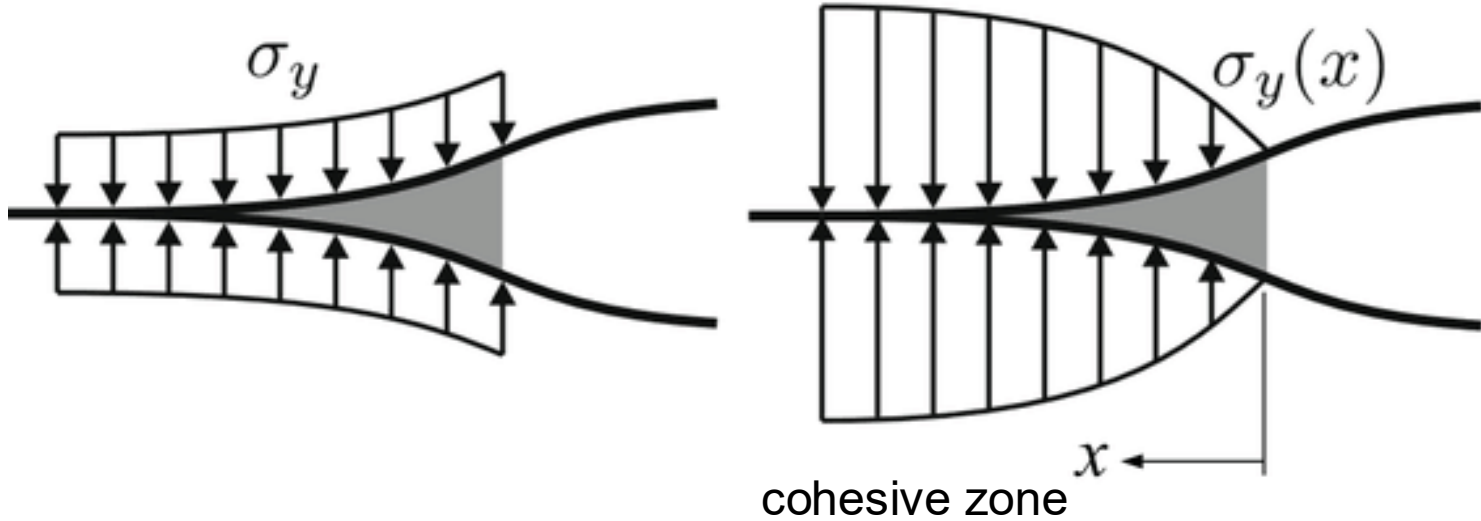
$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} = \sigma^{yield}$$

$$\frac{K_I^2}{2\pi(\sigma^{yield})^2} = r_p^*$$



Ductile fracture

Dugdale-Barenblatt modell



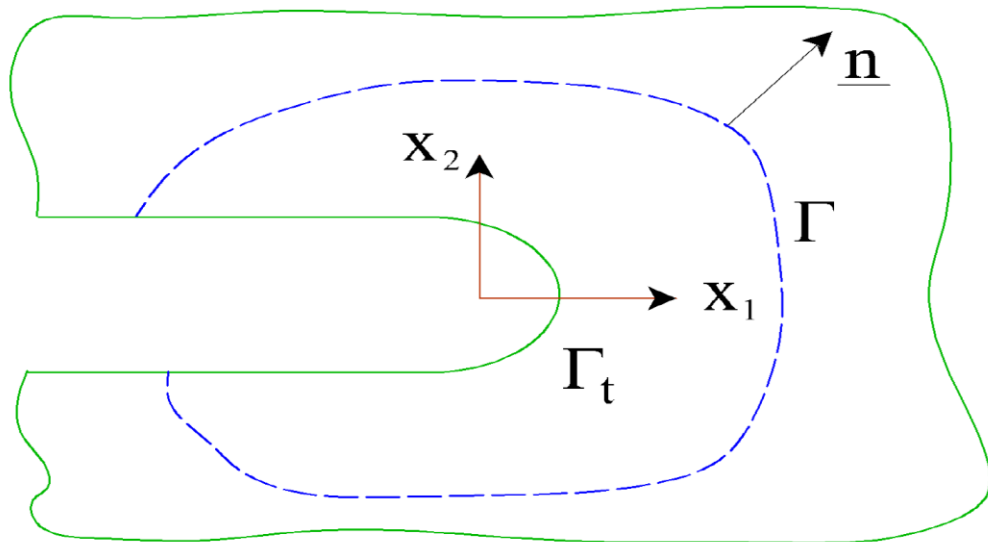
Ductile fracture

Integral J

$$J_i = \int_{\Gamma} \pi n_i - t \frac{\partial u}{\partial x_i} d\Gamma$$

$$\pi = \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij}$$

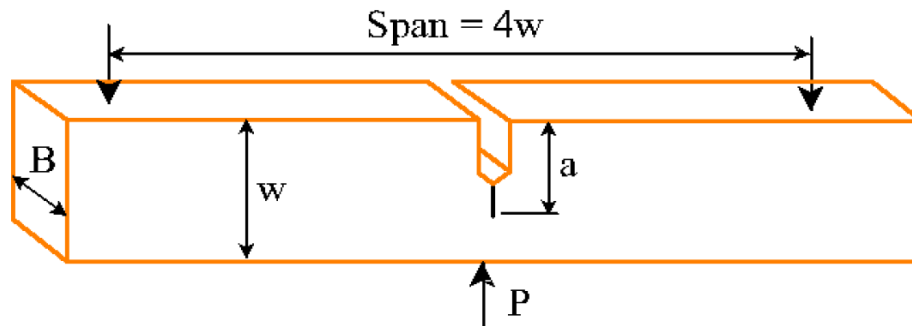
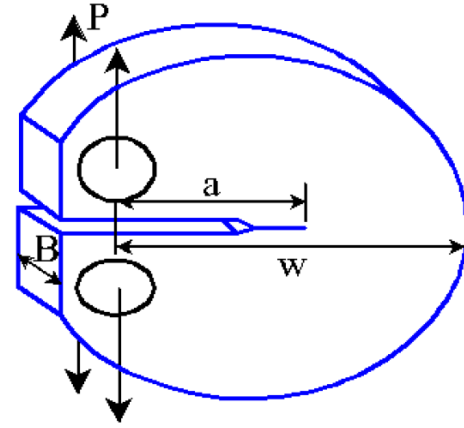
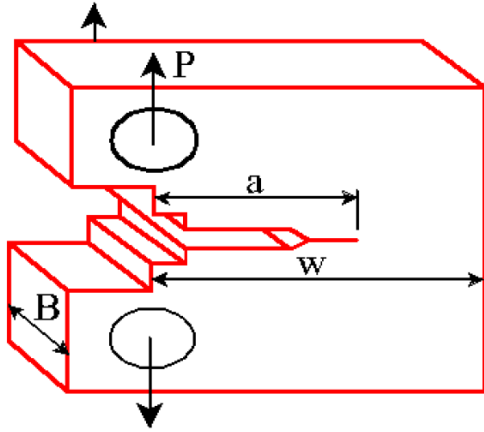
$$\underline{t} = \underline{\sigma} \underline{n}$$



Rice (1968)

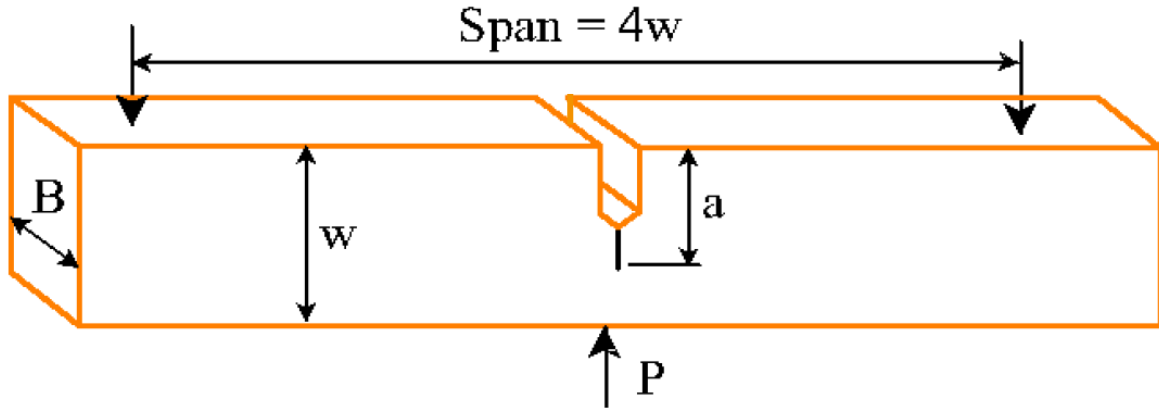


Laboratory identification



Laboratory identification

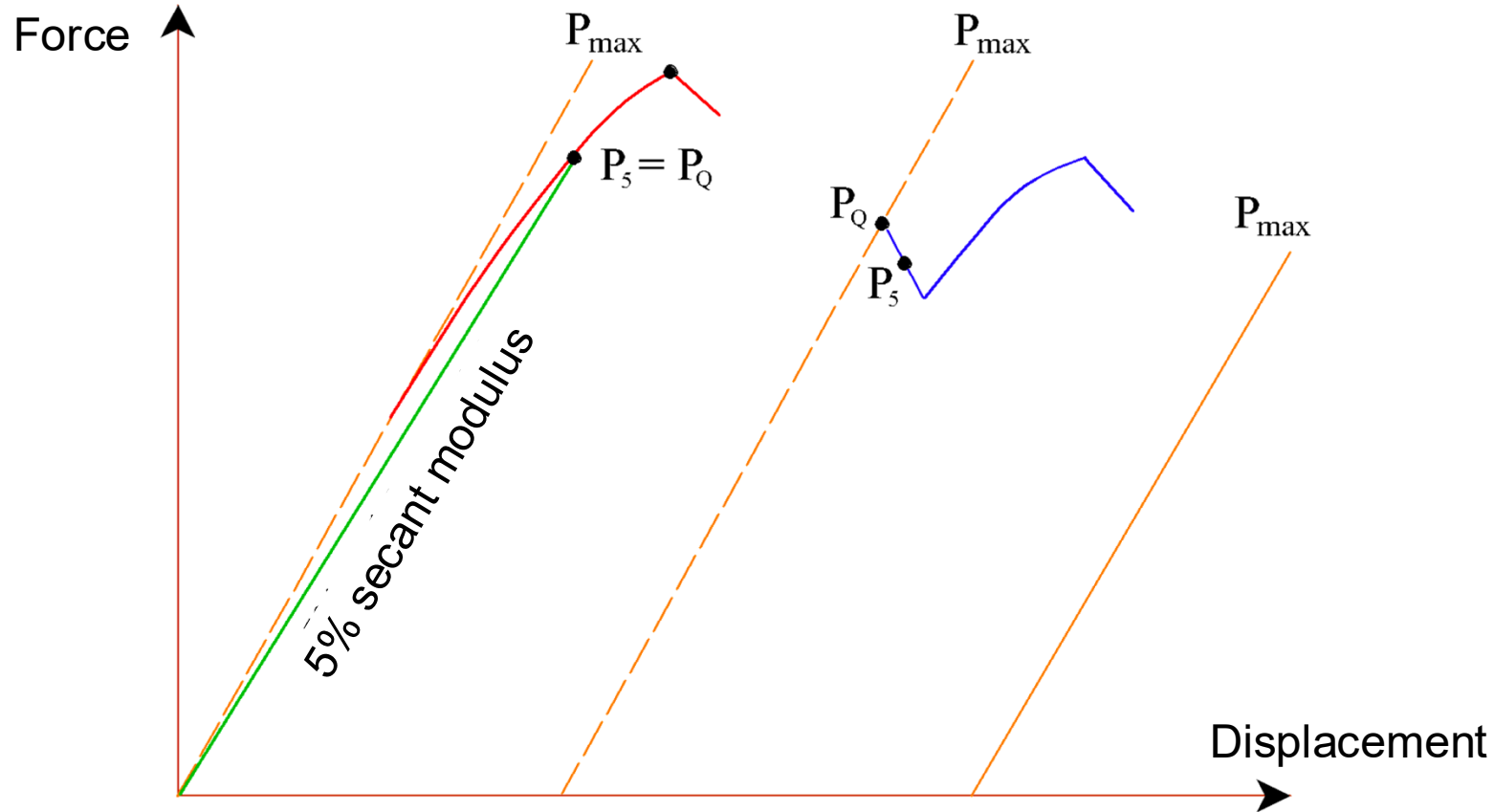
- Crack tip geometry
- Manufacturing
- Sampling



Single-edge notch-bending specimen

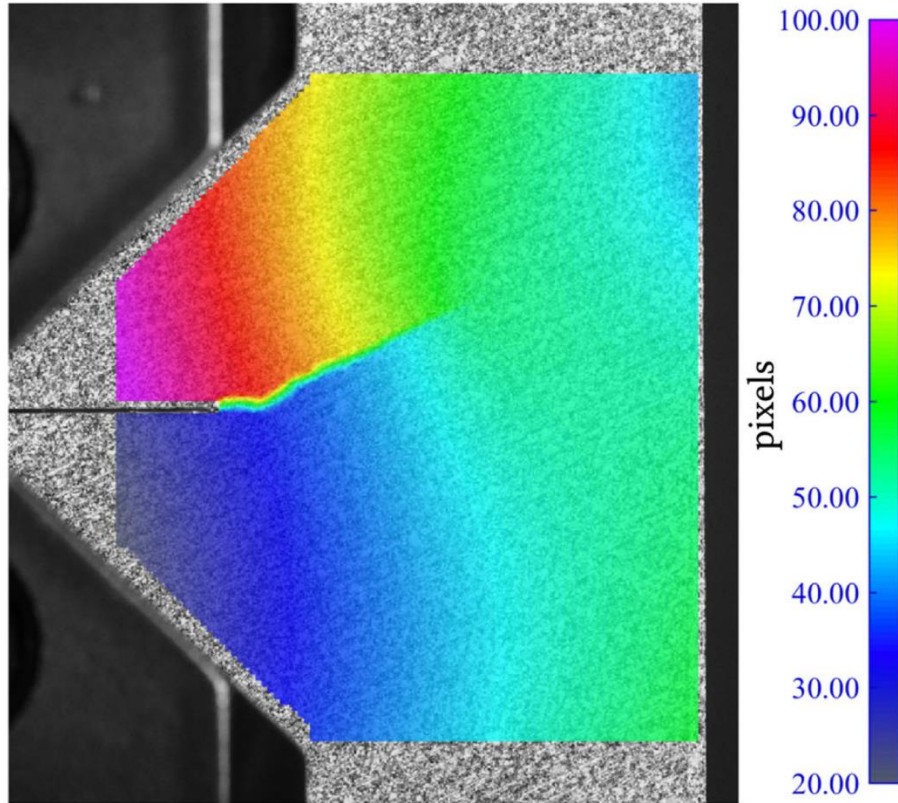
$$K_I = \frac{4P}{B} \sqrt{\frac{\pi}{W}} \left[1.6 \left(\frac{a}{W} \right)^{1/2} - 2.6 \left(\frac{a}{W} \right)^{3/2} + 12.316 \left(\frac{a}{W} \right)^{5/2} \right. \\ \left. - 21.2 \left(\frac{a}{W} \right)^{7/2} + 21.81 \left(\frac{a}{W} \right)^{9/2} \right]$$

Laboratory identification



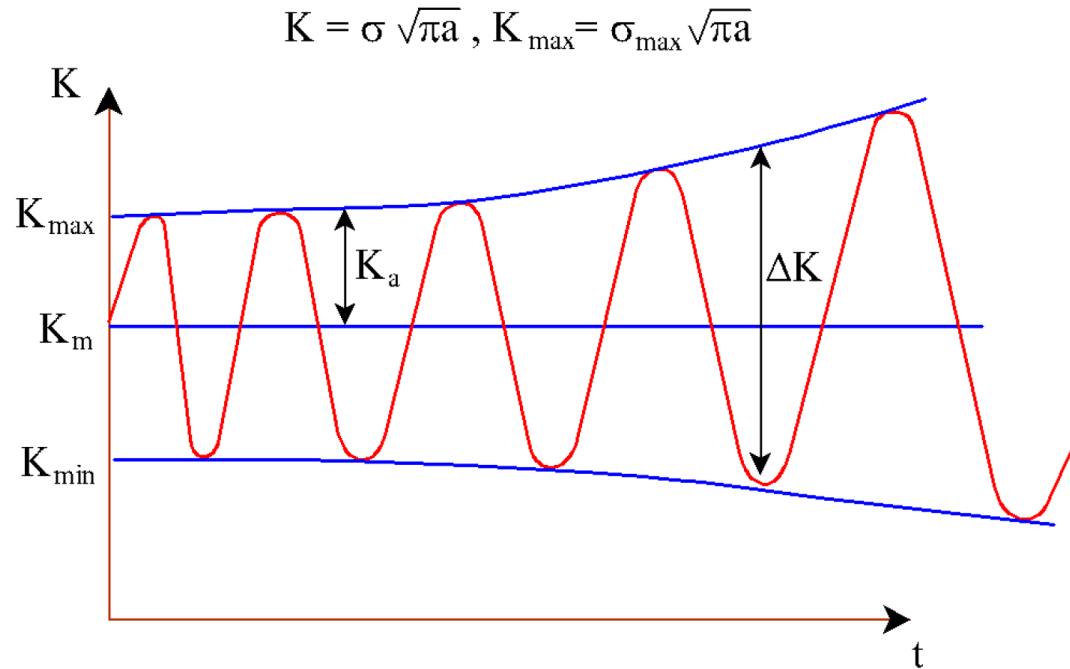
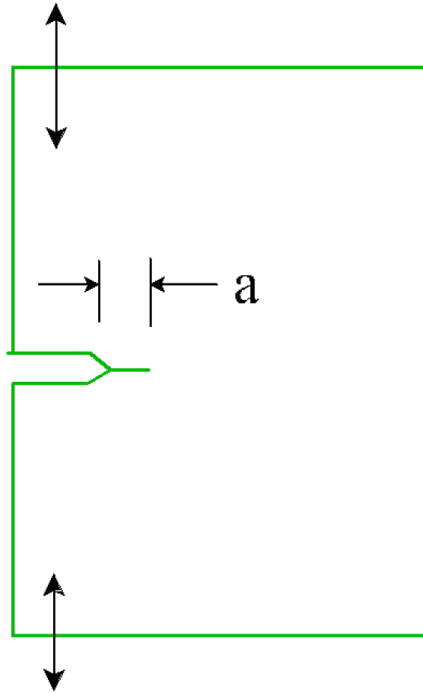
Laboratory identification

Digital Image Correlation



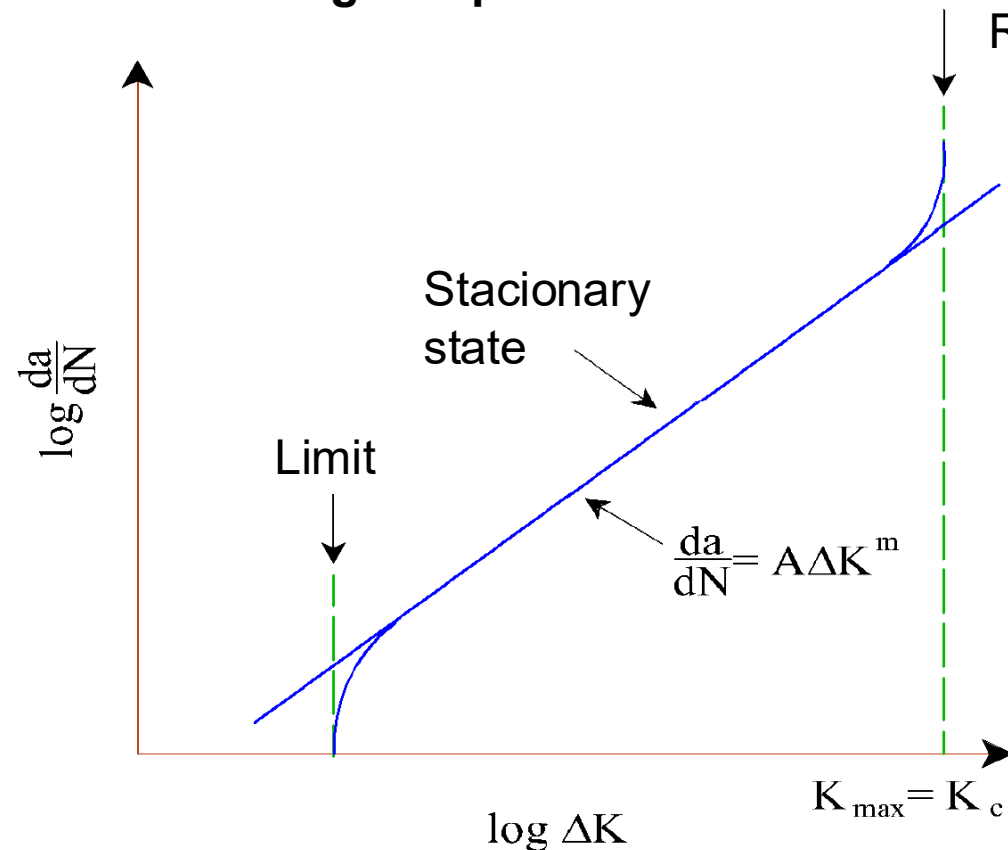
$$u_i(r, \theta) = \sum_{n=1}^{\infty} A_n r^{n/2} g_i^{(n)}(\theta) + \sum_{n=1}^{\infty} B_n r^{n/2} h_i^{(n)}(\theta)$$

Fatigue fracture



Fatigue fracture

Paris-Erdogan equation



Rapid fracture

$$K_I = \sigma_0 \sqrt{a\pi}$$

$$K(a_f) = K_c$$

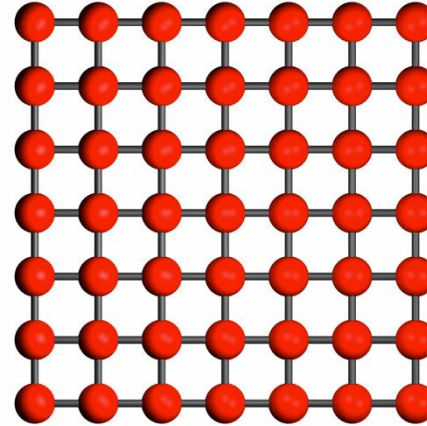
$$\frac{da}{dN} = A(\Delta K)^m$$

$$N_f = \int_0^{N_f} dN = \int_{a_0}^{a_f} \frac{1}{A(\Delta K)^m} da$$

Summary

1. Toughness

(resistance against cracking)



2. Critical stress intensity factor

(critical loading)

$$K_I = \sigma_0 \sqrt{a\pi}$$

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