



Mechanics of deformable solids

Gergely Molnár

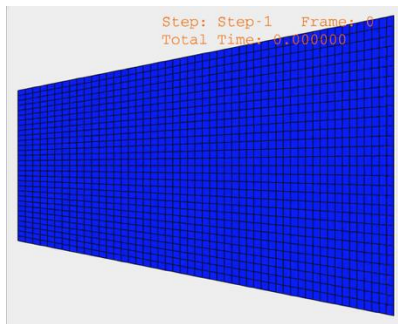


Mechanics of Deformable Solids

Heat Transfer

- DOF: Temperature

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + \dot{q}$$

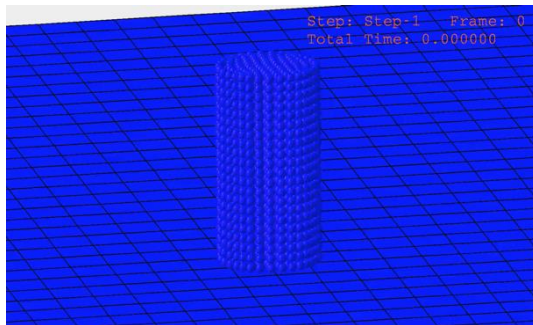


T = 300K

Discrete elements

- DOF: Displacement

$$F = ma$$



Fluid mechanics

- DOF: Velocity

Navier-Stokes equations

DEMONSTRATOR

<https://matthias-research.github.io/pages/tenMinutePhysics/17-fluidSim.html>



Mechanics of Deformable Solids

Equilibrium

$$\nabla \cdot \boldsymbol{\sigma} = 0$$

Compatibility

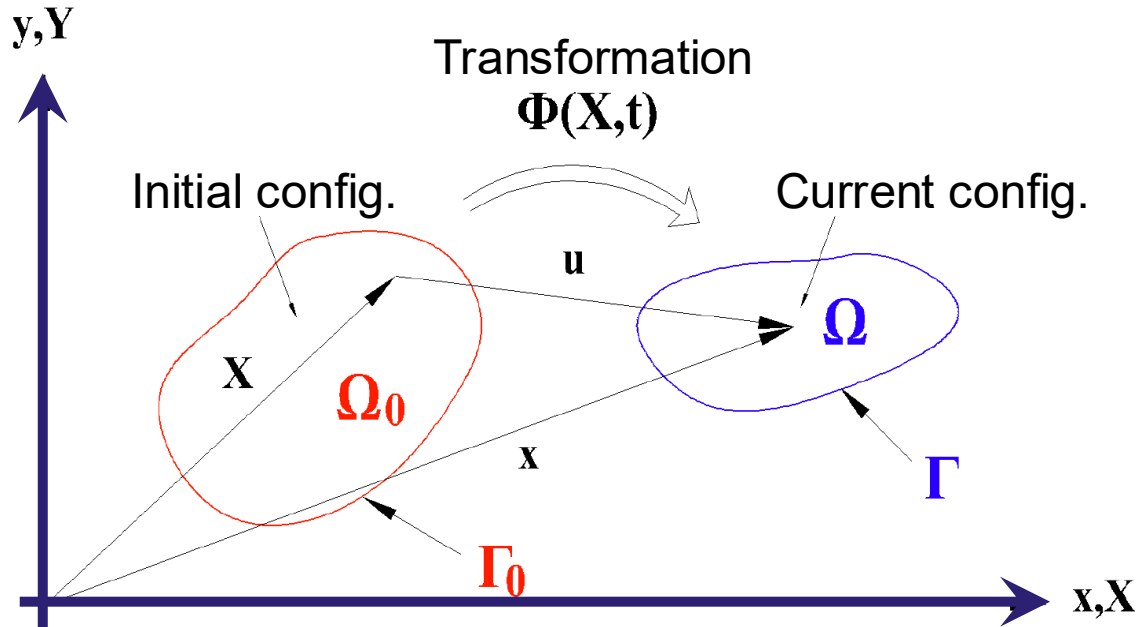
$$\boldsymbol{\epsilon} = G(\boldsymbol{u})$$

Constitutive model

$$\boldsymbol{\sigma} = F(\boldsymbol{\epsilon})$$

Compatibility

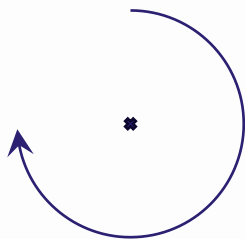
Kinematics



$$x = \Phi(X, t)$$

$$u = x - X$$

Kinematics



$$\mathbf{x} = \Phi(\mathbf{X}, t)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(t\omega) & \sin(t\omega) \\ -\sin(t\omega) & \cos(t\omega) \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

ω angular velocity

$$x = X \cos(t\omega) + Y \sin(t\omega)$$

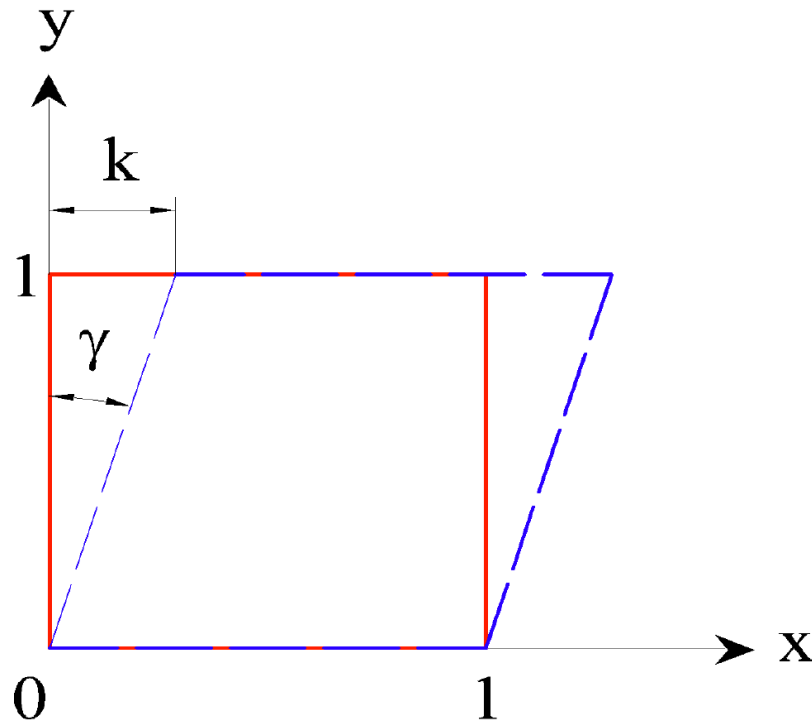
$$y = -X \sin(t\omega) + Y \cos(t\omega)$$

Compatibility

Deformation gradient

$$\mathbf{F} = \frac{\partial \Phi(\mathbf{X}, t)}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \nabla \Phi$$

$$\mathbf{F} = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix}$$



$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \nabla(\mathbf{X} + \mathbf{u}(\mathbf{X}, t)) = \mathbf{I} + \nabla \mathbf{u}$$

Compatibility

Deformation gradient

$$\mathbf{F} = \frac{\partial \Phi(\mathbf{X}, t)}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \nabla \Phi$$

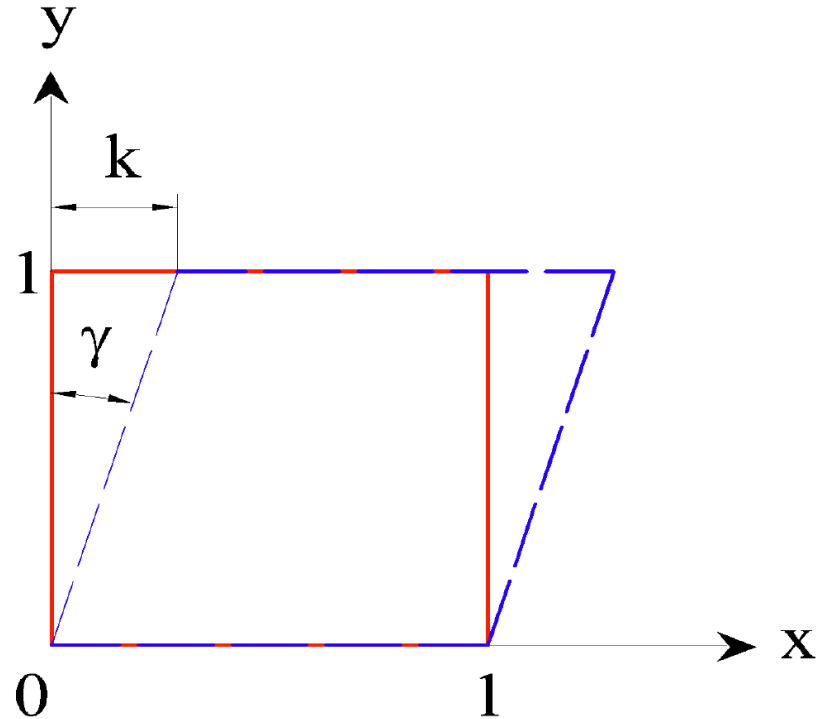
$$x = X + Yk$$

$$y = Y$$

$$z = Z$$



$$\mathbf{F} = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Compatibility

Strain

$$\lambda = \frac{L}{L_0}$$

stretch

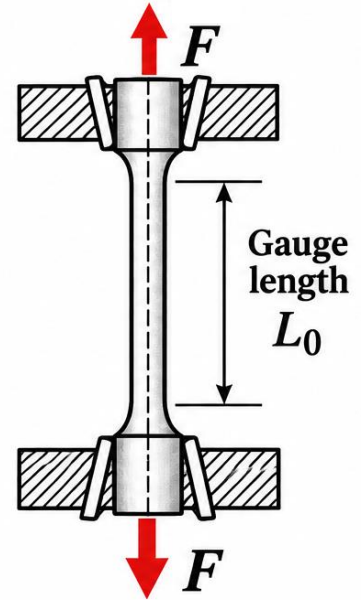
$$\varepsilon = \frac{L - L_0}{L_0} = \frac{\Delta L}{L_0}$$

engineering
(small) strain

$$e = \int_{L_0}^L \frac{dL}{L} = \ln\left(\frac{L}{L_0}\right) = \ln(\lambda)$$

real deformation
(Hencky strain)

$$e = \ln\left(\frac{L}{L_0}\right) = \ln\left(1 + \frac{L - L_0}{L_0}\right) = \ln(1 + \varepsilon)$$



Compatibility

source: Wiki

Volume variation (example for two strains)

$$\epsilon_V = \frac{V - V_0}{V_0} = \frac{\Delta V}{V_0} = \epsilon_x + \epsilon_y + \epsilon_z$$

$$V = [L_0(1 - \epsilon)]^3 \leftarrow \epsilon_x = \epsilon_y = \epsilon_z = \epsilon$$

engineering (small) strain

$$\epsilon = 10^{-3}$$

$$\epsilon = 0.1$$

$$V = 1.003$$

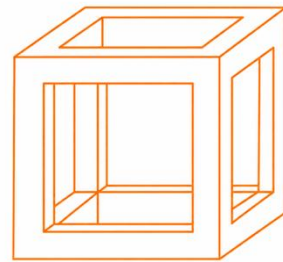
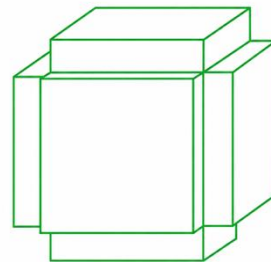
$$V = 1.3310$$

$$\epsilon_V = 0.003$$

$$\epsilon_V = 0.3310$$

$$\epsilon_x + \epsilon_y + \epsilon_z = 0.003$$

$$\epsilon_x + \epsilon_y + \epsilon_z = 0.3$$



Hencky strain

$$e = \ln(1 + \epsilon) = 0.0953$$

$$e_V = \log\left(\frac{V}{V_0}\right) = 0.2859$$

$$e_x + e_y + e_z = 0.2859$$



Strain in 3D

Requirement: under only **rigid-body translation** and/or **rotation**,
strain tensor must be **zero**!

$$\boldsymbol{\epsilon} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] = \frac{1}{2}[\mathbf{F}^T + \mathbf{F}] - \mathbf{I}$$

$$\boldsymbol{\omega} = \frac{1}{2}[\nabla \mathbf{u} - (\nabla \mathbf{u})^T] = \frac{1}{2}[\mathbf{F}^T - \mathbf{F}]$$



Compatibility

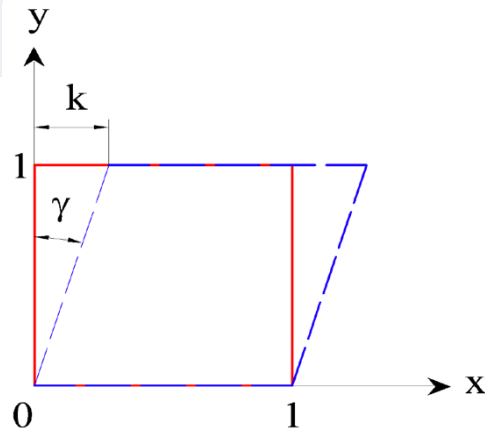
Strain in 3D

Shear example

$$\boldsymbol{\epsilon} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] = \frac{1}{2}[\mathbf{F}^T + \mathbf{F}] - \mathbf{I}$$

$$\boldsymbol{\omega} = \frac{1}{2}[\nabla \mathbf{u} - (\nabla \mathbf{u})^T] = \frac{1}{2}[\mathbf{F}^T - \mathbf{F}]$$

$$\mathbf{F} = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{\epsilon} = \begin{bmatrix} 0 & k/2 & 0 \\ k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \boldsymbol{\omega} = \begin{bmatrix} 0 & -k/2 & 0 \\ k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Compatibility

Strain in 3D

Shear example

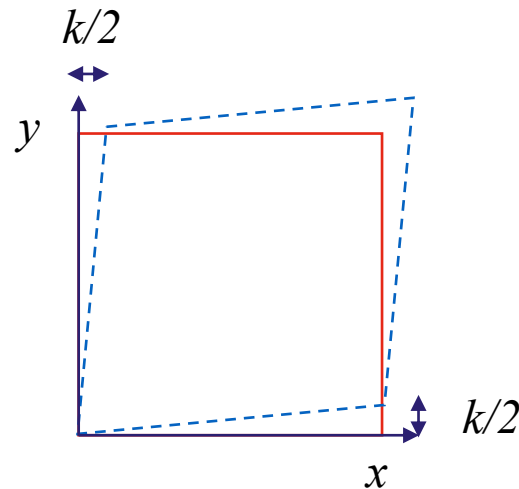
$$\boldsymbol{\epsilon} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] = \frac{1}{2}[\mathbf{F}^T + \mathbf{F}] - \mathbf{I}$$

$$\boldsymbol{\omega} = \frac{1}{2}[\nabla \mathbf{u} - (\nabla \mathbf{u})^T] = \frac{1}{2}[\mathbf{F}^T - \mathbf{F}]$$

$$\mathbf{F} = \begin{bmatrix} 1 & k/2 & 0 \\ k/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{\epsilon} = \begin{bmatrix} 0 & k/2 & 0 \\ k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\omega} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Compatibility

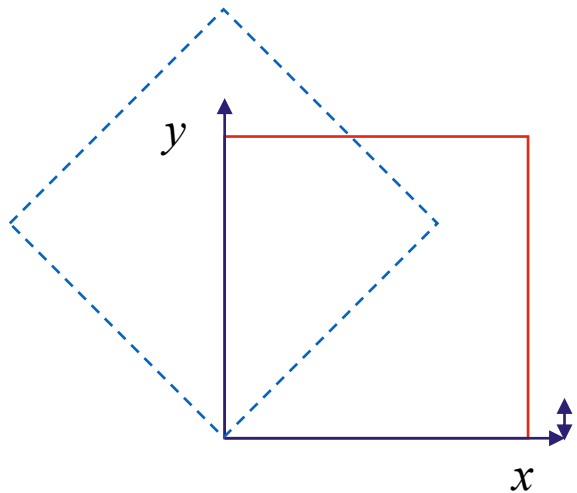
Strain in 3D

Rotation example

$$\mathbf{F} = \frac{\partial \Phi(\mathbf{X}, t)}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \nabla \Phi$$

$$x = X \cos(t\omega) + Y \sin(t\omega)$$

$$y = -X \sin(t\omega) + Y \cos(t\omega)$$



Compatibility

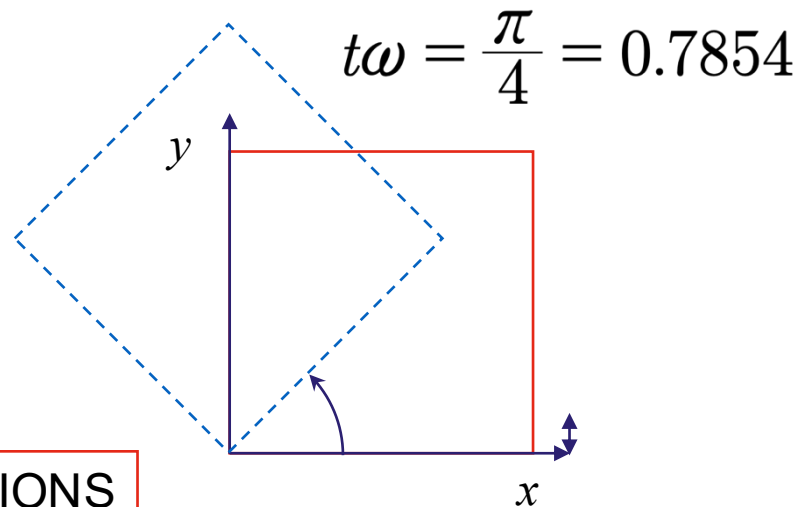
Strain in 3D

Rotation example

$$\mathbf{F} = \begin{bmatrix} \cos(t\omega) & \sin(t\omega) & 0 \\ -\sin(t\omega) & \cos(t\omega) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

LARGE DEFORMATIONS
ARE NECESSARY

$$\boldsymbol{\epsilon} = \begin{bmatrix} -1.2929 & 0 & 0 \\ 0 & -1.2929 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \boldsymbol{\omega} = \begin{bmatrix} 0 & -0.7071 & 0 \\ 0.7071 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Mechanics of Deformable Solids

Equilibrium

$$\nabla \cdot \boldsymbol{\sigma} = 0$$

Compatibility

$$\boldsymbol{\epsilon} = \frac{1}{2} [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$$

Constitutive model

$$\boldsymbol{\sigma} = F(\boldsymbol{\epsilon})$$

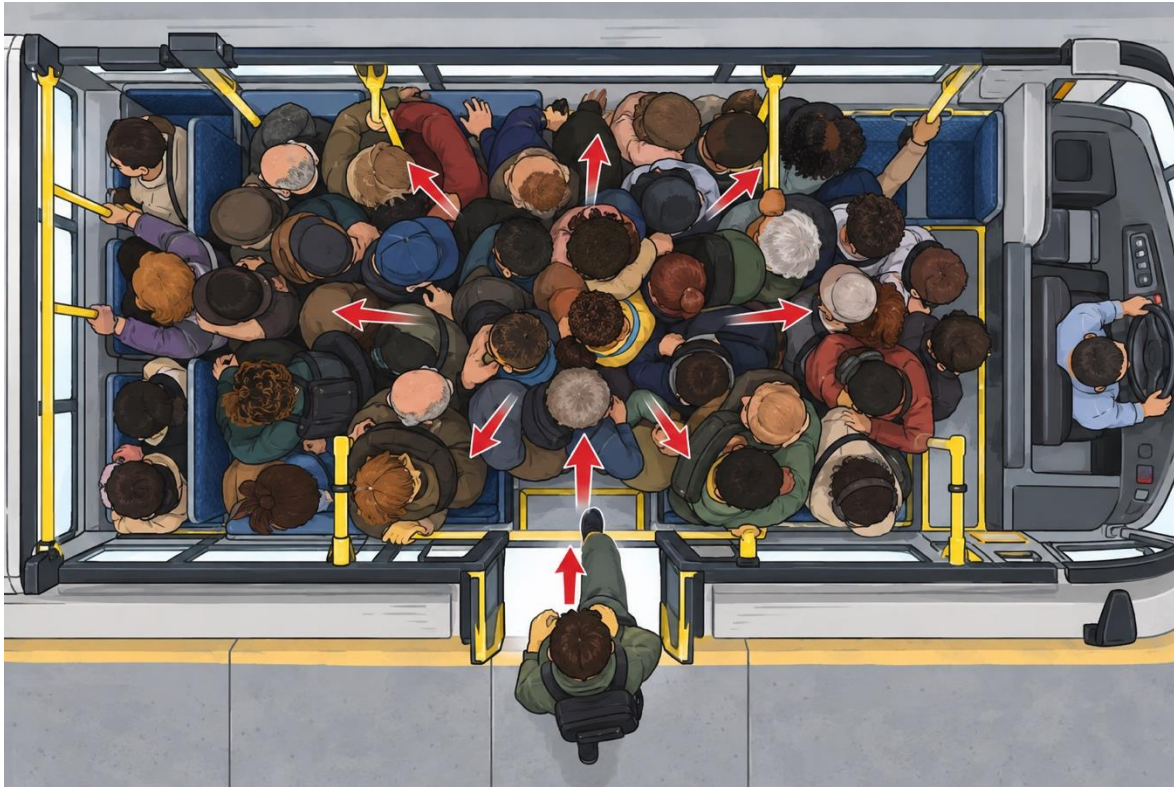
small deformations

small rotations

Equilibrium

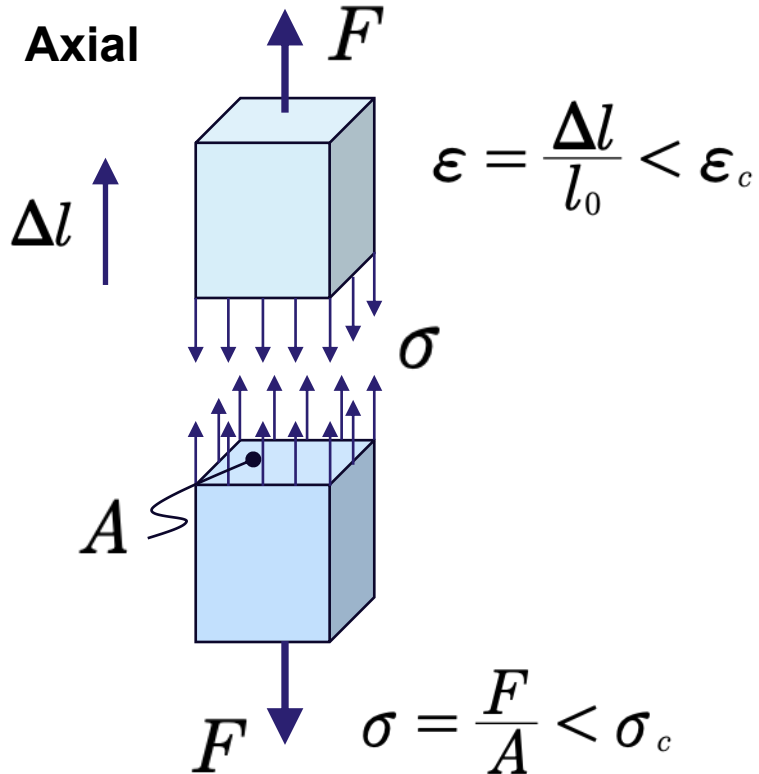
Stress

$$\sigma_c \left[\frac{N}{m^2} \right]$$

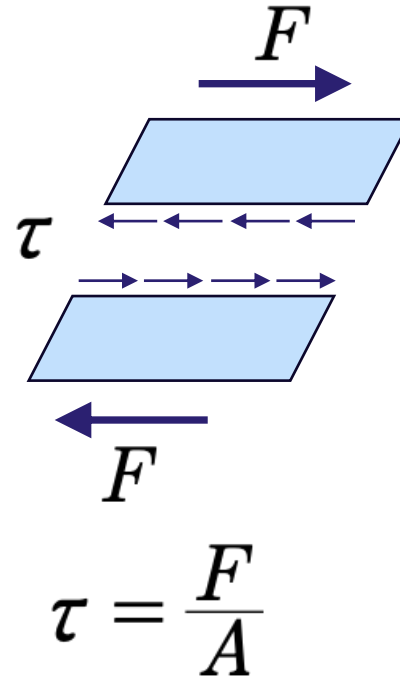


Theoretical basics

Stress

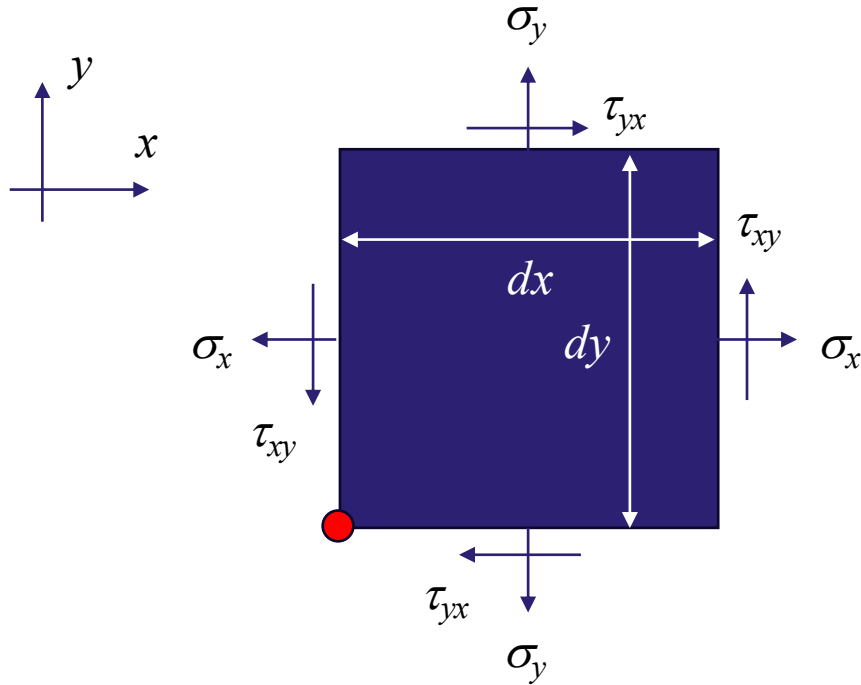


Shear



Equilibrium

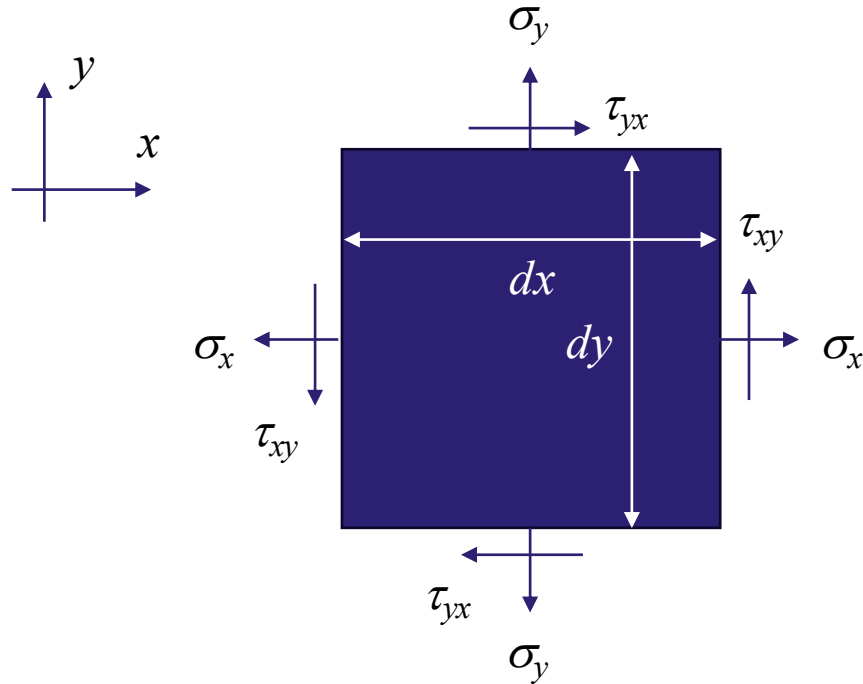
Stress – Tensor



$$\begin{aligned}\sum M &= +\cancel{\sigma_x dy \frac{dy}{2}} - \cancel{\sigma_x dy \frac{dy}{2}} \\ &+ \cancel{\sigma_y dx \frac{dx}{2}} - \cancel{\sigma_y dx \frac{dx}{2}} \\ &+ \tau_{xy} dy \cdot 0 + \tau_{yx} dx \cdot 0 \\ &+ \tau_{xy} dy dx - \tau_{yx} dx dy = 0 \\ \tau_{xy} dy dx &= \tau_{yx} dx dy \\ \tau_{xy} &= \tau_{yx}\end{aligned}$$

Equilibrium

Stress – Tensor



$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} = \mathbf{Q} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \mathbf{Q}^T$$

Equilibrium

Stress – Tensor

Eigenvalues:

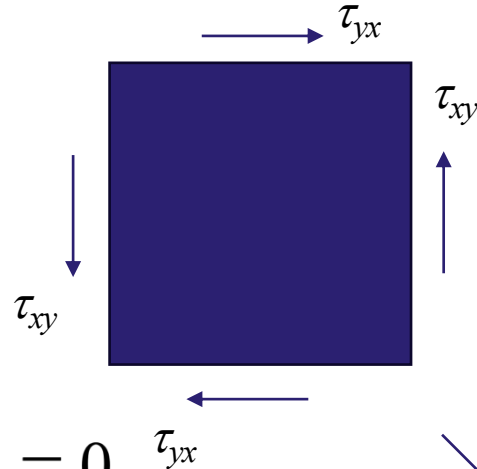
$$\det(\boldsymbol{\sigma} - \sigma_i \mathbf{I}) = 0$$

$$\det(\boldsymbol{\sigma} - \sigma_i \mathbf{I}) = \det\left(\begin{bmatrix} -\sigma_i & \tau_{xy} \\ \tau_{xy} & -\sigma_i \end{bmatrix}\right)$$

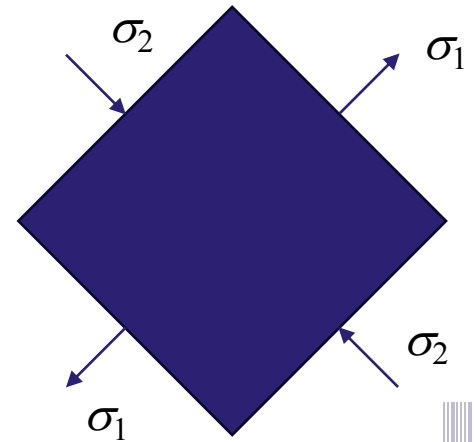
$$\det(\boldsymbol{\sigma} - \sigma_i \mathbf{I}) = (-\sigma_i)(-\sigma_i) - \tau_{xy}\tau_{xy} = 0$$

$$\sigma_i^2 = \tau_{xy}^2$$

$$\hat{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \tau_{xy} & 0 \\ 0 & -\tau_{xy} \end{bmatrix}$$

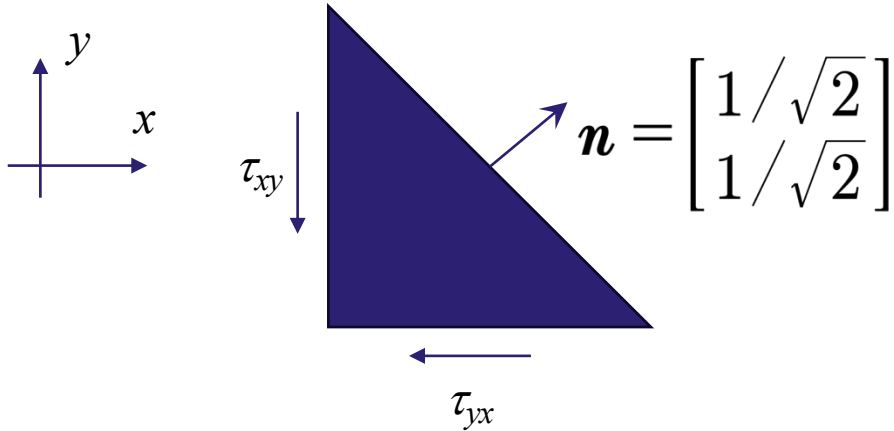


$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \tau_{xy} \\ \tau_{xy} & 0 \end{bmatrix}$$

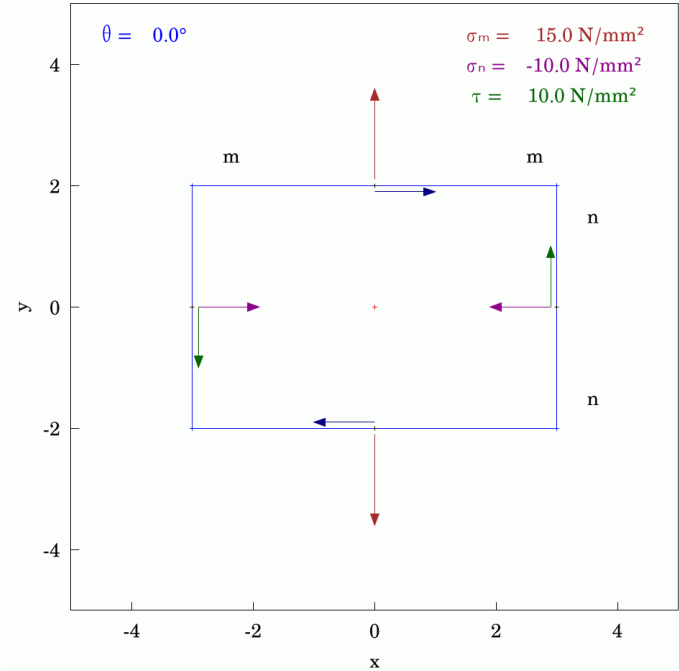


Equilibrium

Stress – Tensor



source: Wikipedia



$$\mathbf{t} = \boldsymbol{\sigma} \mathbf{n} = \begin{bmatrix} 0 & \tau_{xy} \\ \tau_{xy} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \tau_{xy}/\sqrt{2} \\ \tau_{xy}/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \sigma_n \\ \tau_t \end{bmatrix}$$

Equilibrium

$$\nabla \cdot \boldsymbol{\sigma} = 0 \quad ?$$

Airy stress function



$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \quad \begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0 \end{aligned}$$

$$\begin{aligned} \frac{12P}{h^3}y - \frac{3P}{2h}\left(\frac{4 \cdot 2y}{h^2}\right) &= 0 \\ 0 + 0 &= 0 \end{aligned}$$

$$\phi = \frac{3P}{2h} \left(xy - \frac{4xy^3}{3h^2} \right)$$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

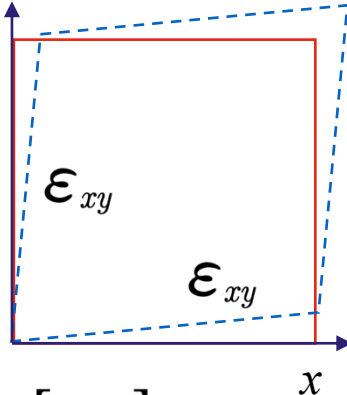
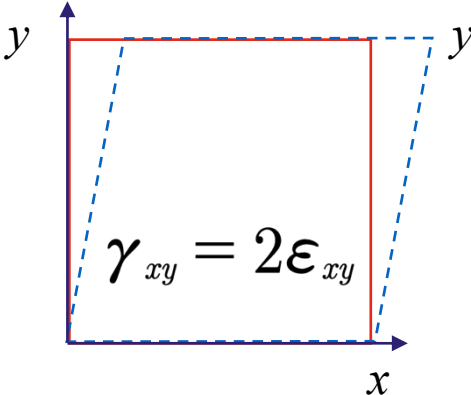
$$\sigma_x = \frac{12P}{h^3}xy = \frac{M}{I_z}y$$

$$\sigma_y = 0$$

$$\tau_{xy} = -\frac{3P}{2h} \left(1 - \frac{4y^2}{3h^2} \right)$$

Equilibrium

Voigt notation

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_x & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_y & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_z \end{bmatrix} \rightarrow \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ 2\epsilon_{xy} \\ 2\epsilon_{xz} \\ 2\epsilon_{yz} \end{bmatrix} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \rightarrow \boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}$$

$$\sigma_c \quad \left[\frac{N}{m^2} \right]$$

Stress - Invariants

- Principal stresses ($\sigma_1, \sigma_2, \sigma_3$)
- Hydrostatic stress / pressure $\sigma_H = \frac{I_1}{3} = \frac{tr(\boldsymbol{\sigma})}{3} = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = -p$
- Deviatoric stress / von Mises stress

$$\sigma_D = \sqrt{J_2} = \sqrt{\frac{1}{6}[(\sigma_x - \sigma_y)^2 + (\sigma_x - \sigma_z)^2 + (\sigma_y - \sigma_z)^2 + 6(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)]} = \frac{\sigma_{vM}}{\sqrt{3}}$$

$$\sigma_D = \sqrt{\frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2]}$$

$$\sigma_c \quad \left[\frac{N}{m^2} \right]$$

Stress - Invariants

- Principal stresses ($\sigma_1, \sigma_2, \sigma_3$)
- Hydrostatic stress / pressure $\sigma_H = \frac{I_1}{3} = \frac{tr(\boldsymbol{\sigma})}{3} = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = -p$
- Deviatoric stress / von Mises stress

$$\sigma_D = \sqrt{J_2} = \sqrt{\frac{1}{6}[(\sigma_x - \sigma_y)^2 + (\sigma_x - \sigma_z)^2 + (\sigma_y - \sigma_z)^2 + 6(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)]} = \frac{\sigma_{vM}}{\sqrt{3}}$$

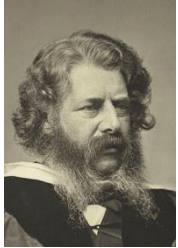
$$\sigma_D = \sqrt{\frac{1}{6}[(\sigma_1 - 0)^2 + (\sigma_1 - 0)^2 + (0 - 0)^2]} = \sqrt{\frac{1}{3}\sigma_1^2} = \sigma_1 \sqrt{\frac{1}{3}} = \frac{\sigma_{vM}}{\sqrt{3}}$$

- Lode angle (ϑ)

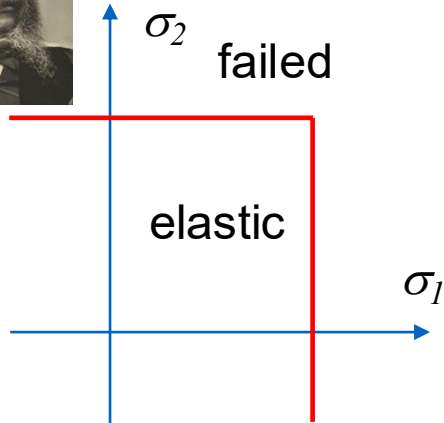
$$\cos(3\vartheta) = \frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}}$$

Equilibrium

Stress - Failure criteria



Rankine

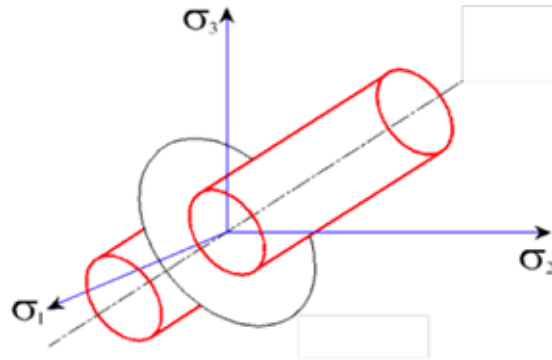


$$f(\boldsymbol{\sigma}, \sigma_c) = \sigma_1 - \sigma_c < 0$$

$$\sigma_1 < \sigma_c$$

$$f(\boldsymbol{\sigma}, \sigma_c) < 0$$

von Mises

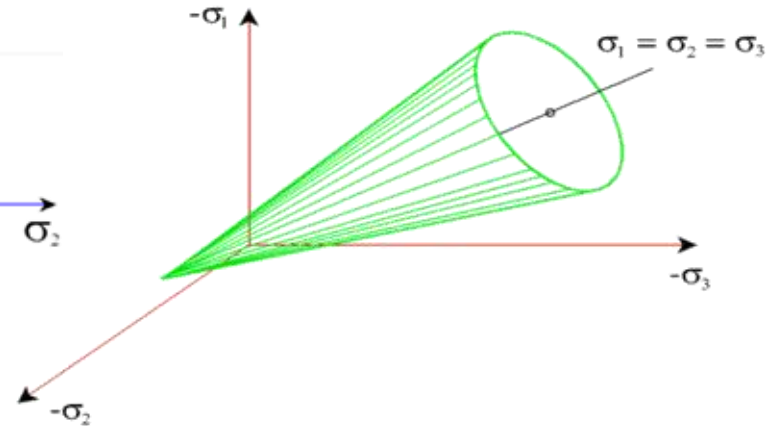


$$f(\boldsymbol{\sigma}, \sigma_c) = \sqrt{3J_2} - \sigma_c < 0$$

$$\sigma_{vM} < \sigma_c$$



Prager-Drucker



$$f(\boldsymbol{\sigma}, \sigma_c) = \sqrt{J_2} - c + \tan(\phi)I_1 < 0$$

Mechanics of Deformable Solids

Equilibrium

$$\nabla \cdot \boldsymbol{\sigma} = 0$$

Compatibility

$$\boldsymbol{\epsilon} = \frac{1}{2} [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$$

Constitutive model

$$\boldsymbol{\sigma} = F(\boldsymbol{\epsilon})$$

small deformations

small rotations

Constitutive equations

Material models

A **material model** describes how a material **responds** to **external influences**.

In engineering:

- **Optical** material model: Light absorption and reflection.
- **Electrical** material model: Electrical conductivity, insulation, and magnetization.
- **Thermal** material model: Thermal conductivity and insulation.

Mechanical Material Model

Describes how a **material responds** to **mechanical loading** and **deformation**.

Constitutive equations

Phenomenological models (Macroscopic)

- A material model is a set of **equations** that **mathematically** describes **experimentally** observed **physical** phenomena.
- Aim: relating **strains**, derived from displacements, to the **stresses** developing within the material.
- To identify the appropriate **conjugate pairs**, we use the **First Law of Thermodynamics** (conservation of energy):

In an **isolated system**, regardless of the processes taking place within it, the **total energy remains constant**.

Constitutive equations

First Law of Thermodynamics

$$\dot{\square} = \frac{\partial \square}{\partial t}$$

In an **isolated system**, regardless of the processes taking place within it, the **total energy remains constant**.

Kinetic External power

$$\dot{K} + \dot{U} = P + Q$$

Internal

Heat

$$K = \frac{1}{2} \int_V \rho \mathbf{v} \cdot \mathbf{v} dV$$

$$K \text{ [J]}$$

kinetic energy

$$\mathbf{v} \left[\frac{\text{m}}{\text{s}} \right]$$

velocity

$$\rho \left[\frac{\text{g}}{\text{m}^3} \right]$$

mass density

$$U = \int_V \rho c dV$$

$$U \text{ [J]} \quad c \left[\frac{\text{J}}{\text{g}} \right]$$

internal energy

energy (mass) density



Constitutive equations

First Law of Thermodynamics

$$\dot{\square} = \frac{\partial \square}{\partial t}$$

In an **isolated system**, regardless of the processes taking place within it, the **total energy remains constant**.

Kinetic External power

$$\dot{K} + \dot{U} = P + Q$$

Internal

Heat

$$P = \int_A \mathbf{t} \cdot \mathbf{v} dA + \int_V \mathbf{b} \cdot \mathbf{v} dV$$

$$P \left[\frac{\text{J}}{\text{s}} \right] \quad \mathbf{t} \left[\frac{\text{N}}{\text{m}^2} \right] \quad \mathbf{b} \left[\frac{\text{N}}{\text{m}^3} \right]$$

external power surface forces volume forces

$$Q = - \int_A \mathbf{q} \cdot \mathbf{n} dA + \int_V \rho r dV$$

$$Q \left[\frac{\text{J}}{\text{s}} \right] \quad \mathbf{q} \left[\frac{\text{J}}{\text{s} \cdot \text{m}^2} \right] \quad r \left[\frac{\text{J}}{\text{g} \cdot \text{s}} \right]$$

heat heat flux vector heat source

Constitutive equations

$$\dot{K} + \dot{U} = P + \dot{Q}$$

First Law of Thermodynamics

$$\frac{\partial}{\partial t}(\rho dV) = 0$$



$$\dot{K} = \frac{1}{2} \int_V \left[\frac{\partial}{\partial t}(\rho) v \cdot v + \rho \frac{\partial}{\partial t}(v \cdot v) \right] dV = \frac{1}{2} \int_V [\rho(\dot{v} \cdot v + v \cdot \dot{v})] dV$$

$$\nabla \cdot (\mathbf{A}^T \mathbf{u}) = \nabla \cdot \mathbf{A} \cdot \mathbf{u} + \mathbf{A} : \nabla \mathbf{u}$$

$$= \int_V [\rho v \cdot \dot{v}] dV = \int_V [v \cdot (\nabla \cdot \boldsymbol{\sigma} + \mathbf{b})] dV$$

$$\rho \dot{v} = \rho \mathbf{a} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{b}$$

$$= \int_V [\nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{v}) - \boldsymbol{\sigma} : \nabla \mathbf{v} + \mathbf{b} \cdot \mathbf{v}] dV$$

$$\boldsymbol{\sigma} : \nabla \mathbf{v} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} + \cancel{\boldsymbol{\sigma} : \dot{\boldsymbol{\omega}}}$$



Constitutive equations

$$\dot{K} + \dot{U} = P + Q$$

First Law of Thermodynamics

$$\dot{U} = \frac{\partial}{\partial t} \int_V \rho c dV = \int_V \left[\frac{\partial \rho}{\partial t} c + \rho \dot{c} \right] dV = \int_V \rho \dot{c} dV$$

$$P = \int_A \boldsymbol{\sigma} \cdot \mathbf{n} \cdot \mathbf{v} dA + \int_V \mathbf{b} \cdot \mathbf{v} dV = \int_V [\nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{v}) + \mathbf{b} \cdot \mathbf{v}] dV$$

$$\int_A \mathbf{A} \cdot \mathbf{n} dA = \int_V \nabla \cdot \mathbf{A} dV$$
$$\int_A \mathbf{u} \cdot \mathbf{n} dA = \int_V \nabla \cdot \mathbf{u} dV$$

divergence
theorem

$$Q = - \int_A \mathbf{q} \cdot \mathbf{n} dA + \int_V \rho r dV = \int_V [-\nabla \cdot \mathbf{q} + \rho r] dV$$

$$\int_V [\rho \dot{c} + \nabla \cdot \mathbf{q} - \rho r - \boldsymbol{\sigma} : \nabla \mathbf{v}] dV = 0 \quad \longrightarrow \quad \dot{c} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} + \rho r - \nabla \cdot \mathbf{q}$$

Constitutive equations

Directionality of Processes

- **Second Law of Thermodynamics**
- **Mechanical processes** through **changes in entropy**:
 - **Reversible processes** — e.g. elastic deformation → **entropy remains constant**
 - **Irreversible processes** — e.g. plastic deformation, damage, or fracture → **entropy increases**.
- **Clausius–Duhem inequality:**
$$\Delta S = S_2 - S_1 \geq \int_{t_1}^{t_2} \frac{d\bar{Q}}{T} dt \quad \left[\frac{J}{K} \right]$$

Constitutive equations

Second Law of Thermodynamics

the total entropy of an isolated system, a measure of disorder or **unusable energy**, must **always increase** or **remain constant** over time

$$\dot{S} = \frac{\partial}{\partial t} \int_V \rho \eta dV = \int_V \left[-\nabla \cdot \left(\frac{\mathbf{q}}{T} \right) + \frac{\rho r}{T} \right] dV \quad \eta \quad \left[\frac{J}{gK} \right]$$

$$\int_V \left[\rho \dot{\eta} + \nabla \cdot \left(\frac{\mathbf{q}}{T} \right) - \frac{\rho r}{T} \right] dV \geq 0 \longrightarrow \rho \dot{\eta} + \nabla \cdot \left(\frac{\mathbf{q}}{T} \right) - \frac{\rho r}{T} \geq 0$$

$$\rho \dot{\eta} - \frac{\rho r}{T} + \frac{1}{T} \nabla \cdot \mathbf{q} - \frac{1}{T^2} \mathbf{q} \cdot \nabla T \geq 0 \longrightarrow \rho \dot{\eta} - \frac{\rho r}{T} + \frac{1}{T} \nabla \cdot \mathbf{q} \geq 0$$

$\uparrow$
 $\mathbf{q} \cdot \nabla T \leq 0$

Constitutive equations

Criteria

1. **Coordinate invariance**

$$\mathbf{r}' = \mathbf{Q} \cdot \mathbf{r} + \mathbf{f}$$

2. **History dependence**

$$\mathbf{c}' = \mathbf{c}$$

3. Principle of **locality**

$$\mathbf{u}' = \mathbf{Q} \cdot \mathbf{u}$$

4. Principle of **simultaneity**

$$\boldsymbol{\sigma} = \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T$$

5. Material **objectivity**

$$\mathbf{F}' = \mathbf{Q} \cdot \mathbf{F}$$

6. Compatibility with the **fundamental equations of physics**

Constitutive equations

Compatibility with the **fundamental equations of physics**

$$\left. \begin{aligned} \dot{\rho} &= \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} + \rho r - \nabla \cdot \mathbf{q} \\ \rho \dot{\eta} T &= \rho r - \nabla \cdot \mathbf{q} \end{aligned} \right\} \begin{aligned} \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} &= \rho(\dot{c} - \dot{\eta} T) & \psi &= \rho(c - \eta T) \\ \frac{\partial \psi}{\partial t} &= \rho(\dot{c} - \dot{\eta} T - \eta \dot{T}) & \psi &= \psi(\mathbf{X}, \boldsymbol{\epsilon}, T, \nabla T, \dots) \\ & & & \text{Helmholtz free energy} \end{aligned}$$

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - \dot{\psi} + \rho \eta \dot{T} \geq 0$$

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - \left[\frac{\partial \psi}{\partial \boldsymbol{\epsilon}} \frac{\partial \boldsymbol{\epsilon}}{\partial t} + \frac{\partial \psi}{\partial T} \frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial \nabla T} \frac{\partial \nabla T}{\partial t} \right] - \rho \eta \dot{T} \geq 0$$

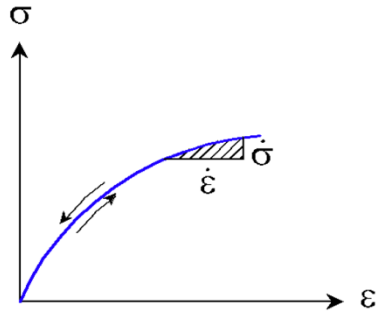
$$\left(\boldsymbol{\sigma} - \frac{\partial \psi}{\partial \boldsymbol{\epsilon}} \right) : \dot{\boldsymbol{\epsilon}} + \left(-\rho \eta - \frac{\partial \psi}{\partial T} \right) \dot{T} - \frac{\partial \psi}{\partial \nabla T} \nabla \dot{T} \geq 0$$

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\epsilon}}$$

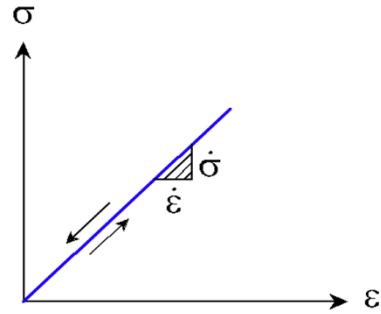
Constitutive equations

Postulate of Drucker

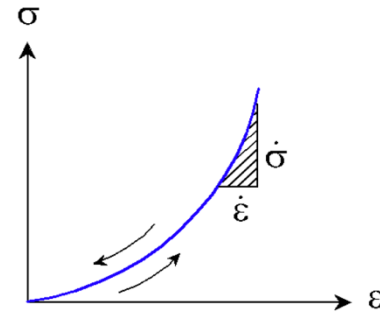
Elastic materials



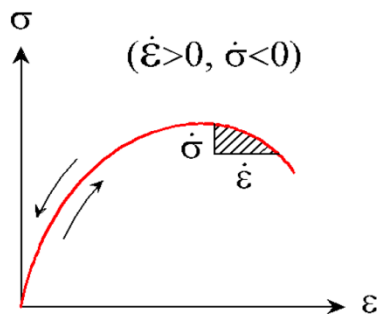
(a)



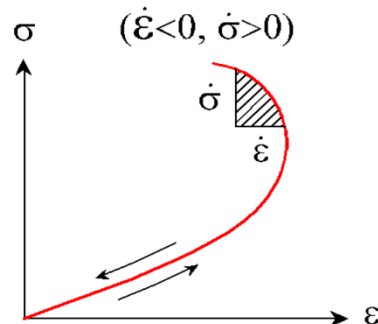
(b)



(c)



(d)



(e)

$$\underline{\dot{\sigma}}^T \underline{\dot{\epsilon}} \geq 0$$

$$\underline{\dot{\sigma}}^T (\underline{\dot{\epsilon}} - \underline{\dot{\epsilon}}_{el}) \geq 0$$

Constitutive equations

Postulate of Drucker

Elastic materials

$$\underline{\dot{\sigma}} = \frac{\partial \underline{\sigma}}{\partial \underline{\epsilon}} \underline{\dot{\epsilon}} = \frac{\partial^2 \psi}{\partial \underline{\epsilon} \partial \underline{\epsilon}} \underline{\dot{\epsilon}} \quad \underline{\dot{\sigma}}^T \underline{\dot{\epsilon}} \geq 0 \implies \left(\frac{\partial^2 \psi}{\partial \underline{\epsilon} \partial \underline{\epsilon}} \underline{\dot{\epsilon}} \right)^T \underline{\dot{\epsilon}} \implies (\mathbb{C} \underline{\dot{\epsilon}})^T \underline{\dot{\epsilon}}$$

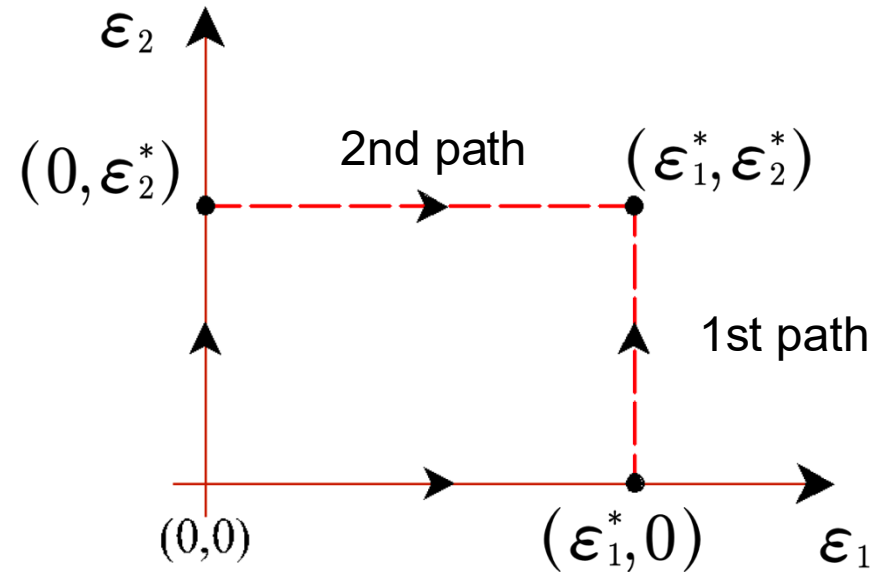
$$\mathbb{C} = \begin{bmatrix} \frac{\partial^2 \psi}{\partial \epsilon_x^2} & \frac{\partial^2 \psi}{\partial \epsilon_x \partial \epsilon_y} & \frac{\partial^2 \psi}{\partial \epsilon_x \partial \epsilon_z} & \frac{\partial^2 \psi}{\partial \epsilon_x \partial \gamma_{xy}} & \frac{\partial^2 \psi}{\partial \epsilon_x \partial \gamma_{xz}} & \frac{\partial^2 \psi}{\partial \epsilon_x \partial \gamma_{yz}} \\ & \frac{\partial^2 \psi}{\partial \epsilon_y^2} & & & & \\ & & \frac{\partial^2 \psi}{\partial \epsilon_z^2} & & \dots & \\ & & & \frac{\partial^2 \psi}{\partial \gamma_{xy}^2} & & \\ & sym & & & \frac{\partial^2 \psi}{\partial \gamma_{xz}^2} & \\ & & & & & \frac{\partial^2 \psi}{\partial \gamma_{yz}^2} \end{bmatrix}$$

Constitutive equations

Symmetry

$$\pi^{(1)} = \int_{(0,0)}^{(\varepsilon_1^*, 0)} (\sigma_1 d\varepsilon_1 + \sigma_2 d\varepsilon_2) + \int_{(\varepsilon_1^*, 0)}^{(\varepsilon_1^*, \varepsilon_2^*)} (\sigma_1 d\varepsilon_1 + \sigma_2 d\varepsilon_2) \leftarrow \pi = \int_0^{\varepsilon_i} \sigma_i d\varepsilon_i$$

$$c_{12} \neq c_{21} \quad \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$



Constitutive equations

Symmetry

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$$

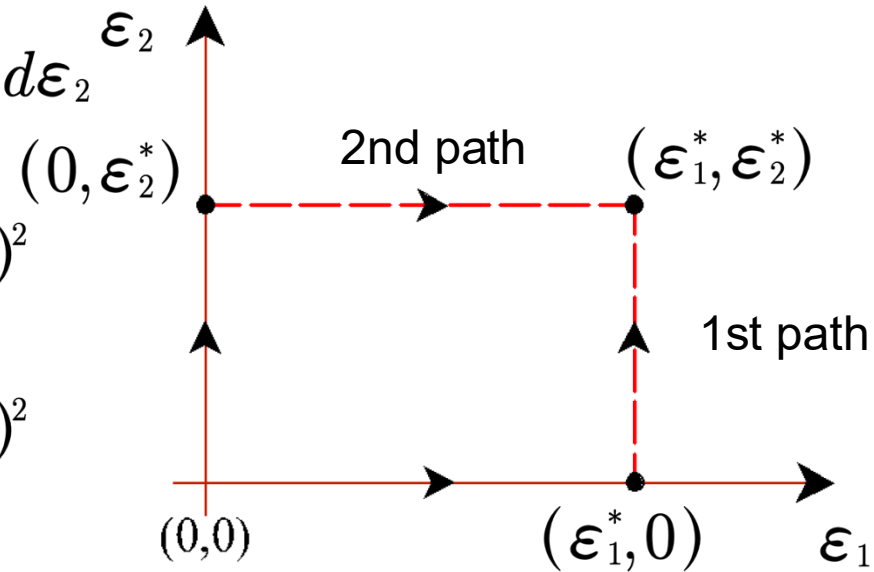
$$\pi^{(1)} = \int_{(0,0)}^{(\epsilon_1^*, 0)} (\sigma_1 d\epsilon_1 + \sigma_2 d\epsilon_2) + \int_{(\epsilon_1^*, 0)}^{(\epsilon_1^*, \epsilon_2^*)} (\sigma_1 d\epsilon_1 + \sigma_2 d\epsilon_2)$$

$$\pi^{(1)} = \int_0^{\epsilon_1^*} c_{11} \epsilon_1 d\epsilon_1 + \int_0^{\epsilon_2^*} (c_{21} \epsilon_1^* + c_{22} \epsilon_2) d\epsilon_2$$

$$\pi^{(1)} = \frac{1}{2} c_{11} (\epsilon_1^*)^2 + c_{21} \epsilon_1^* \epsilon_2^* + \frac{1}{2} c_{22} (\epsilon_2^*)^2$$

$$\pi^{(2)} = \frac{1}{2} c_{11} (\epsilon_1^*)^2 + c_{12} \epsilon_1^* \epsilon_2^* + \frac{1}{2} c_{22} (\epsilon_2^*)^2$$

$$\text{if } c_{12} \neq c_{21} \implies \pi^{(1)} \neq \pi^{(2)}$$



Constitutive equations

Hooke's law

- **Linear elastic** materials
- Rigidity is **independent** of **deformation**

$$\mathbb{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}$$

sym

isotrope

$$\mathbb{C} = C \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$C = \frac{E}{(1+\nu)(1-2\nu)}$$

Constitutive equations

Hooke's law

- **Linear elastic** materials
- Rigidity is **independent** of **deformation**

$$\mathbb{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \quad \mathbb{C} = \begin{bmatrix} 2\mu + \lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu + \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu + \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

sym isotrope

Constitutive equations

Hooke's law

- **Linear elastic** materials
- Rigidity is **independent** of **deformation**

isotrope

$$\mathbb{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}$$

sym

$$\psi = \sum_{i=1}^3 \mu \varepsilon_i^2 + \frac{\lambda}{2} \text{tr}(\boldsymbol{\varepsilon})^2$$

Constitutive equations

Plane strain vs Plane stress (2D)

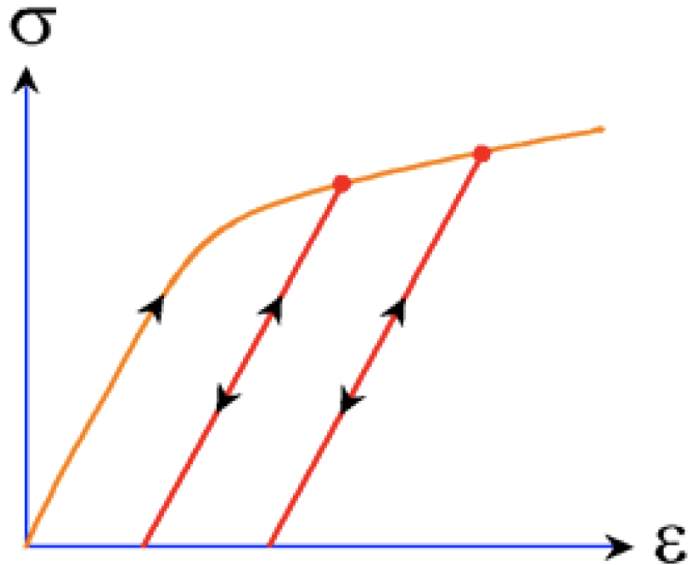
$$\mathbb{C}^\varepsilon = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \Rightarrow \sigma_z = \nu(\sigma_x + \sigma_y)$$

$$\mathbb{C}^\sigma = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \Rightarrow \varepsilon_z = -\frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_y)$$

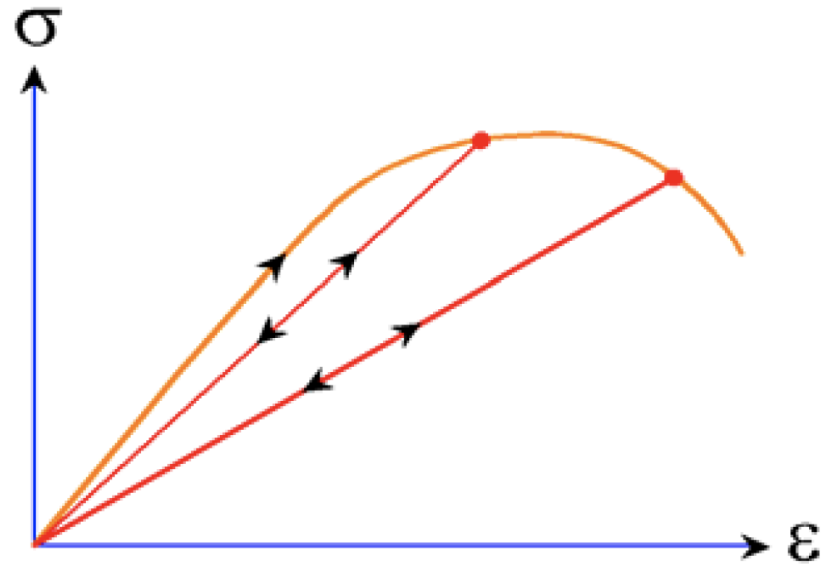


Constitutive equations

Dissipative models



Plasticity



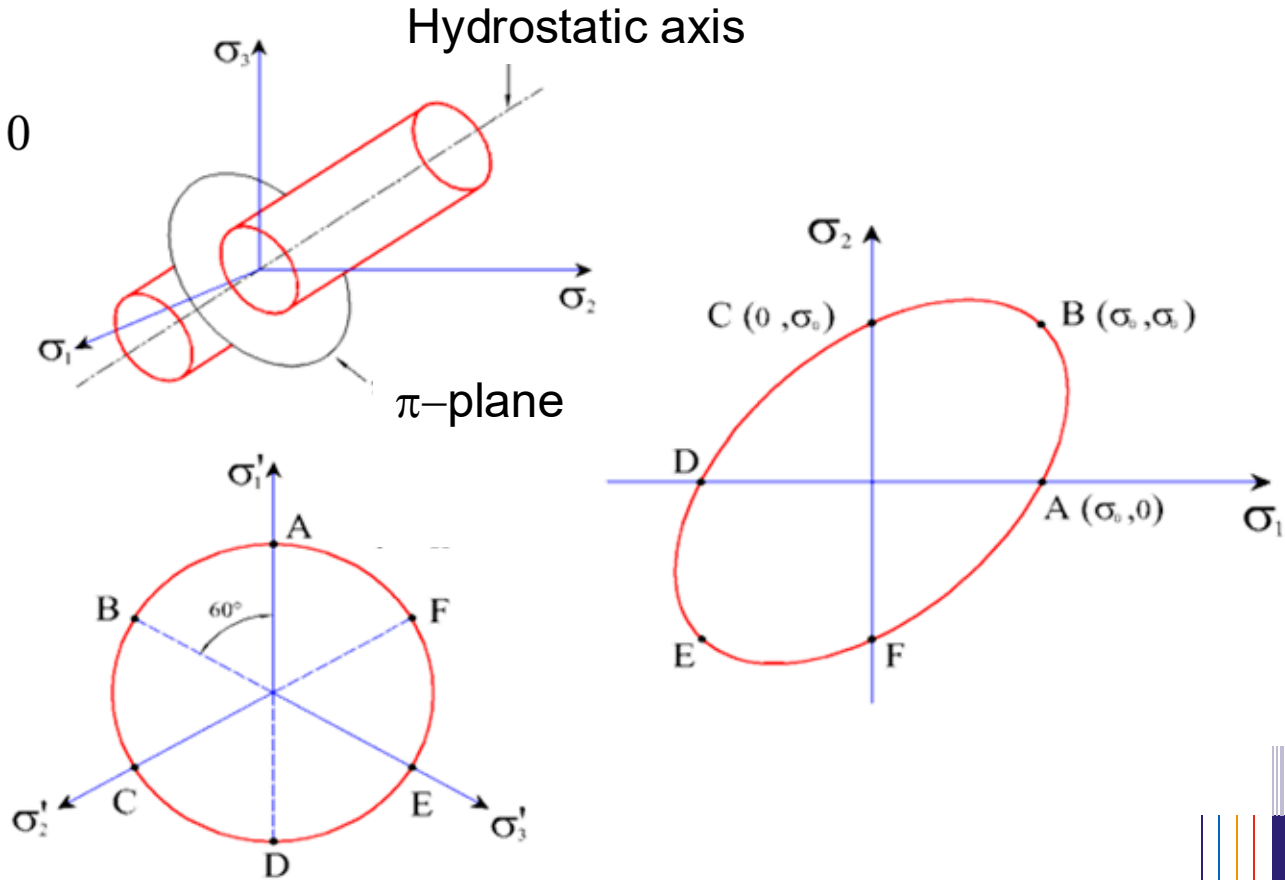
Damage

Constitutive equations

Ductile models

$$f(\boldsymbol{\sigma}, \sigma_c) = \sqrt{3J_2} - \sigma_c < 0$$

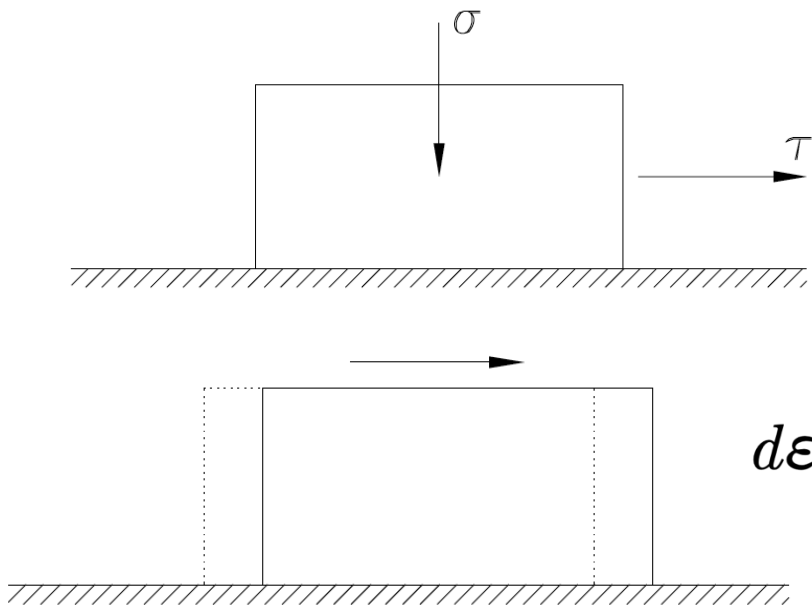
$$\sigma_{vM} < \sigma_c$$



Constitutive equations

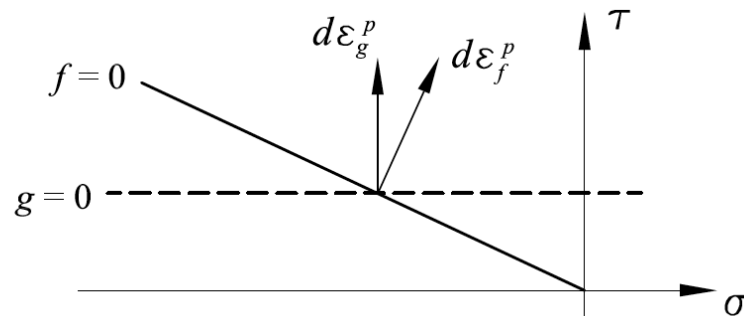
Ductile models

Associate vs non-associate models



$$f = \tau - \hat{\mu}\sigma$$

$$g = \tau$$

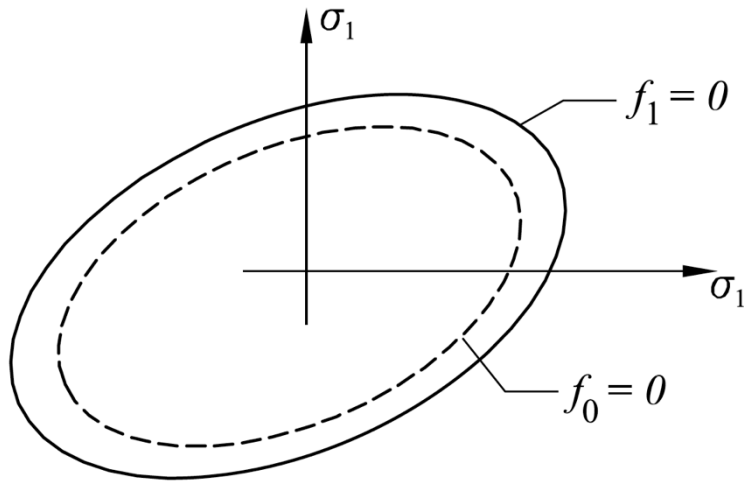
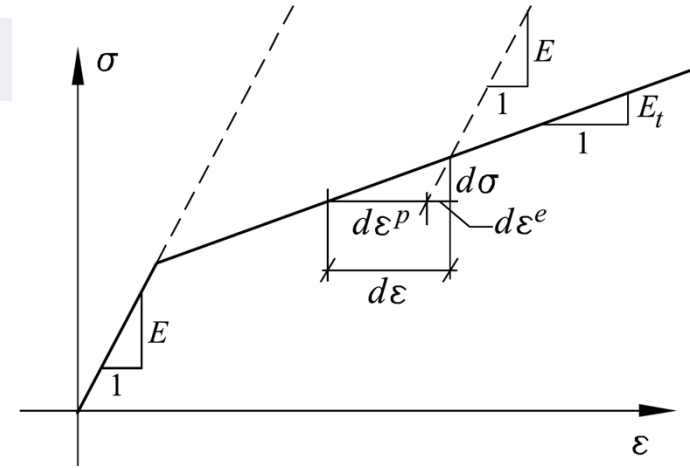


$$d\epsilon_{pl} = d\hat{\lambda} \nabla_{\sigma} g = d\hat{\lambda} \begin{bmatrix} \frac{\partial g}{\partial \sigma} \\ \frac{\partial g}{\partial \tau} \end{bmatrix} = d\hat{\lambda} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

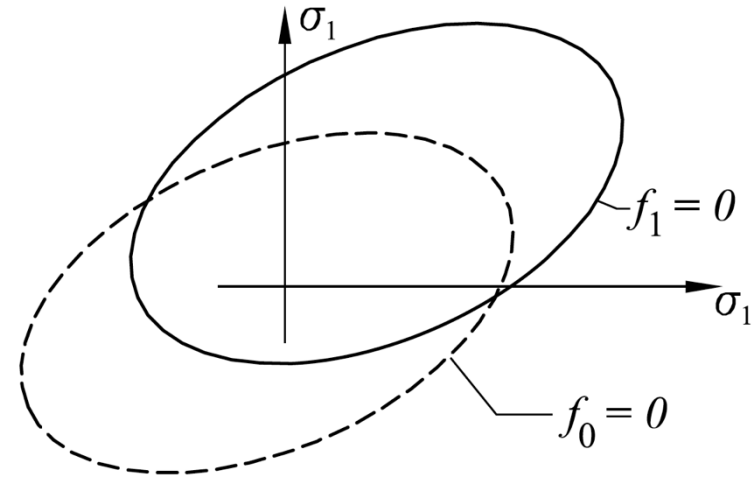
Constitutive equations

Hardening

$$f(\boldsymbol{\sigma}) - \sigma_c(\alpha) = 0$$



Isotropic



Kinematic

Constitutive equations

Damage models

L. M. Kachanov (1958)



$$\psi = \frac{1}{2}(1 - D) \underline{\underline{\boldsymbol{\varepsilon}}}^T \underline{\underline{\mathbb{C}}} \underline{\underline{\boldsymbol{\varepsilon}}}$$

$$\dot{\eta} = \underline{\underline{\boldsymbol{\sigma}}}^T \underline{\underline{\dot{\boldsymbol{\varepsilon}}}} - \dot{\psi} \geq 0 \quad \text{Clausius - Duhem}$$

$$\underline{\underline{\boldsymbol{\sigma}}}^T \underline{\underline{\dot{\boldsymbol{\varepsilon}}}} - \frac{\partial \psi}{\partial \underline{\underline{\boldsymbol{\varepsilon}}}} \underline{\underline{\dot{\boldsymbol{\varepsilon}}}} - \frac{\partial \psi}{\partial D} \dot{D} \geq 0$$

damage driving
force

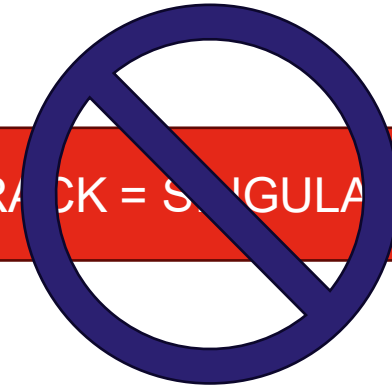
$$\underline{\underline{\boldsymbol{\sigma}}} = \frac{\partial \psi}{\partial \underline{\underline{\boldsymbol{\varepsilon}}}} \quad -\frac{\partial \psi}{\partial D} \dot{D} \geq 0$$

$$Y = \frac{\partial \psi}{\partial D}$$

$$f(\underline{\underline{\boldsymbol{\varepsilon}}}, Y, D) = 0$$

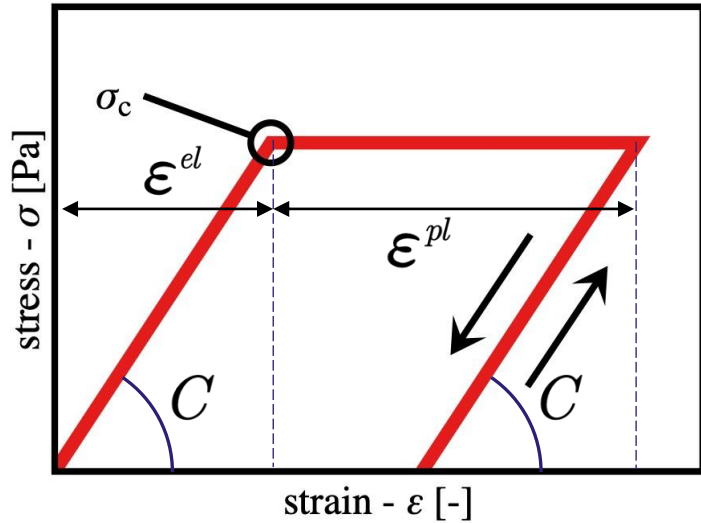
$$dD = -d\hat{\lambda} \frac{\partial f}{\partial Y}$$

CRACK = SINGULARITY



Constitutive equations

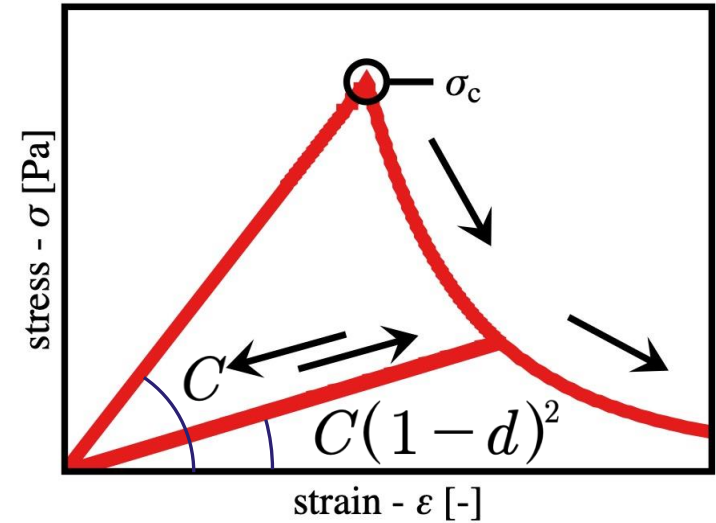
Plasticity



$$\sigma = \epsilon^{el} C$$

$$\epsilon = \epsilon^{el} + \epsilon^{pl}$$

Damage

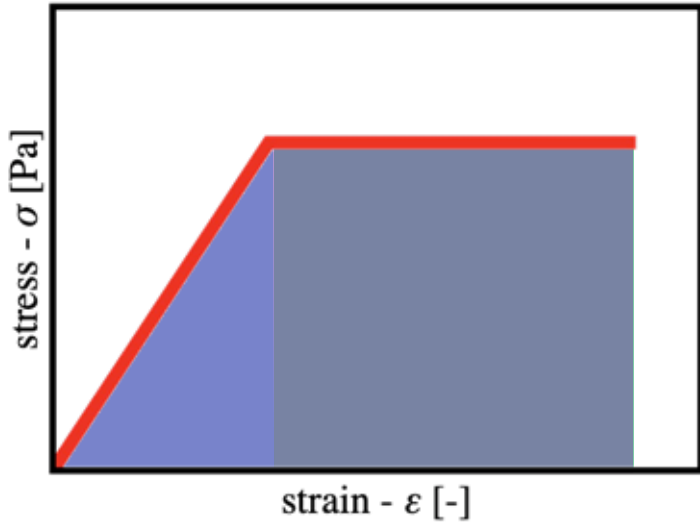


$$\sigma = \epsilon C (1 - d)^2$$

$$d = f(\epsilon, \sigma, \psi, \dots)$$

Constitutive equations

Energy / Stress / Rigidity



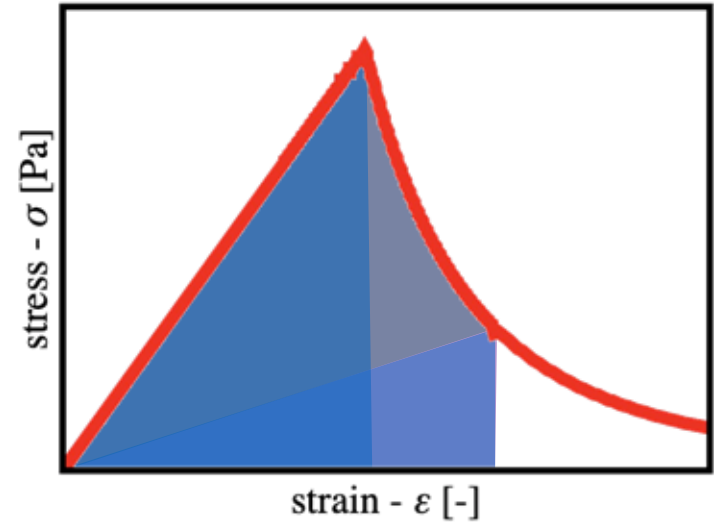
$$w_{ext} = \psi_{el} + \psi_{diss}$$

$$\psi_{el} = \frac{\sigma \epsilon^{el}}{2} = \frac{1}{2} C (\epsilon^{el})^2 \quad \psi_{diss} \rightarrow \psi_{pl} = \int \sigma \dot{\epsilon}^{pl} dt$$

$$\sigma = \frac{d\psi_{el}}{d\epsilon^{el}}$$

$$w_{ext} = \int \sigma \dot{\epsilon} dt$$

Damage



$$\psi_{el} = \frac{\sigma(1-d)^2 \epsilon}{2} = \frac{1}{2} (1-d)^2 C (\epsilon^{el})^2$$

Conclusion

Mechanics of deformable solids

- Compatibility $\nabla \times \boldsymbol{F} = \mathbf{0}$
- Equilibrium $\rho \boldsymbol{a} = \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b}$
- Constitutive relations
 - Energy conservation (LTD1) $\dot{K} + \dot{U} = P + Q$
 - Increasing dissipation (LTD2) $\dot{\eta} = \underline{\sigma}^T \dot{\underline{\varepsilon}} - \dot{\psi} \geq 0$

References

1. Kachanov, L. M. (1958). Time of the rupture process under creep conditions. *Izvestiya Akademii Nauk SSSR, Otdelenie Tekhnicheskikh Nauk* [*Proceedings of the USSR Academy of Sciences, Division of Engineering Sciences*], **8**, 26–31.
2. Bojtár, I. *Nemlineáris kontinuummechanika* [*Nonlinear Continuum Mechanics*]. Lecture notes, Budapest University of Technology and Economics, Budapest.
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